Logic and its Applications to the Sciences

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Logical languages and semantics can be used to help us reason about problems in science. We will focus on three examples of such problems:

- Quantum theory
- Computation and time
- Social behavior
Quantum Theory
**Locality**: A property, assumed by classical (as opposed to quantum) physics “that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past.”


**Bell inequalities**: Certain inequalities (due to Bell’s theorem) that are derivable from the assumption of classical physics. If Bell inequalities are violated, then so are the assumptions of classical physics.

**Logical Bell inequalities**: A logical formulation of Bell inequalities that covers a wide range of non-local behaviors.

Alain Aspect style experiment

30° \rightleftharpoons \text{switch} \rightarrow \text{Florescence} \rightarrow \text{switch} \rightarrow 60°

\downarrow

0° \rightleftharpoons \text{coincidence monitor} \rightarrow 30°
## Experiment results

<table>
<thead>
<tr>
<th></th>
<th>$(B, B)$</th>
<th>$(P, B)$</th>
<th>$(B, P)$</th>
<th>$(P, P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(30°, 30°)$</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>$(30°, 60°)$</td>
<td>3/8</td>
<td>1/8</td>
<td>1/8</td>
<td>3/8</td>
</tr>
<tr>
<td>$(0°, 30°)$</td>
<td>3/8</td>
<td>1/8</td>
<td>1/8</td>
<td>3/8</td>
</tr>
<tr>
<td>$(0°, 60°)$</td>
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<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

- $P$ - passes through
- $B$ - blocked

The numbers represent the fraction of pairs of photons with the behavior given by the column.
**Game**


- You and your partner start in a room (the Calcium atom exhibiting florescence) and make a plan
- You go to separate rooms (and can no longer interact with each other)
- Each of you is told an angle and then asked whether you pass or will be blocked:
- These previous three steps are repeated an arbitrary number of times

Goal is to have guesses match the experimental data with the repeated plays.

Classical (non-quantum) physics assumes particles must interact only by local means, thus conforming to the goal of this game.
Consider the set of “atomic propositions”:

\[
\text{AtProp} = \{ L30, L0, R30, R60 \}
\]

Each round of the game, you and your partner must say for some \( p \in \text{AtProp} \) whether \( p \) is “true” (blocking) or “false” (passing).

The set of propositional formulas is the smallest set satisfying:

- Any \( p \in \text{AtProp} \) is a formula.
- If \( \varphi \) is a formula, then \( \neg \varphi \) (“not \( \varphi \)”) is a formula.
- If \( \varphi, \psi \) are formulas, then \( \varphi \land \psi \) (“and”), \( \varphi \lor \psi \) (“or”), \( \varphi \rightarrow \psi \) (“only if”) are formulas.

After determining which atomic propositions are true, we can determine for each formula whether it is true.
Example

Suppose $V : \text{AtProp} \rightarrow \{t, f\}$, such that

- $V(L0) = f$
- $V(L30) = f$
- $V(R30) = t$
- $V(R60) = t$

From this, we can extend $v$ to truth values for other formulas, such as

- $V(\neg R30) = f$
- $V(L0 \land R30) = f$
- $V(L0 \lor R30) = t$
Let

- $\varphi_1, \ldots, \varphi_n$ be formulas of propositional logic.
- $p_i$ for $1 \leq i \leq n$ each be the probability of $\varphi_i$ being true.

If the $\varphi_i$ are jointly contradictory ($V(\bigvee_i \neg \varphi_i) = t$ for every $V$), then

$$1 = \Pr(\bigvee_i \neg \varphi_i) \leq \sum_i \Pr(\neg \varphi_i) = \sum_i (1 - p_i) = n - \sum_i p_i.$$ 

Thus

$$\sum_i p_i \leq n - 1$$
Consider

\[ \varphi_1 = (L30 \land R30) \lor (\neg L30 \land \neg R30) \quad p_1 = 1 \]
\[ \varphi_2 = (L30 \land R60) \lor (\neg L30 \land \neg R60) \quad p_2 = 3/4 \]
\[ \varphi_3 = (L0 \land R30) \lor (\neg L0 \land \neg R30) \quad p_3 = 3/4 \]
\[ \varphi_4 = (L0 \land \neg R60) \lor (\neg L0 \land R60) \quad p_4 = 3/4 \]

For all assignment of truth values to \( \text{AtProp} \), \( \bigwedge_i \varphi_i \) is false (jointly contradictory). But \( \sum_i p_i = 3 + 1/4 > 3 = n - 1 \)
Quantum theory supports the following (each row still adds to one, but the constraints are looser):

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>$(a, b)$</td>
<td>$&gt; 0$</td>
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<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>$(a', b)$</td>
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<td>$&gt; 0$</td>
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</tr>
<tr>
<td>$(a, b')$</td>
<td>$= 0$</td>
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$$
\varphi_1 = a \land b \quad p_1 > 0
$$

$$
\varphi_2 = \neg(a' \land b) \quad p_2 = 1
$$

$$
\varphi_3 = \neg(a \land b') \quad p_3 = 1
$$

$$
\varphi_4 = a' \lor b' \quad p_4 = 1
$$

The $\varphi_i$ are jointly contradictory. But $\sum_i p_i > 3 = n - 1$. 
Computation and Time
Five philosophers sit around a round dining table, with a big bowl of rice in the middle. There is one chopstick to the right of each philosopher.
Possible states of a philosopher:

- $e$ eating rice (but must have two chopsticks in hand)
- $t$ thinking (but must have no chopsticks in hand)
- $w_r$ waiting for right chopstick
  (must hold left chopstick, but not right)
- $w_l$ waiting for left chopstick
  (must hold right chopstick, but not left)
- $r_r$ returning right chopstick
  (must hold right chopstick, but not left)
- $r_l$ returning left chopstick
  (must hold left chopstick, but not right)

A global state is a tuple $(s_1, s_2, s_3, s_4, s_5)$ of local states for each philosopher, that is compatible with the constraints above.

**Example**

$(t, e, w_l, t, t)$ could be a global state, but $(e, e, t, t, t)$ does not satisfy the constraints.
The following actions can be performed of each chopstick

- $p_l$ being picked up by philosopher to the left
- $p_r$ being picked up by philosopher to the right
- $t_l$ being returned by philosopher to the left
- $t_r$ being returned by philosopher to the right
- $\epsilon$ no action

A concurrent action is a tuple of local actions:

**Example**

$(p_r, p_r, p_r, p_r, p_r)$ is the concurrent action where every philosopher picks up the chopstick to her left.
Philosophers as computational processes

Suppose each philosopher follows a strict protocol. For example, if a philosopher picks up one chopstick, she will wait until the other is available before doing anything else.

- **Deadlock** occurs if no further concurrent actions are possible. Example: after everyone picks up the chopstick on their right and waits indefinitely for the one on the left (no further concurrent actions are possible).

- **Fairness** every philosopher may have a chance to eat infinitely often (fair allocation of computational resources to the processes)
A Labelled transition system (with properties) is a tuple

\[ M = (S, \text{Act}, \text{AtProp}, \{ R_a \}_{a \in \text{Act}}, V) \]

- \( S \) is a set of (global) states
- \( \text{AtProp} \) set of atomic properties:
  - Example: \( e_3 \) (third philosopher is eating).
- \( \text{Act} \) set of (concurrent) actions:
- \( R_a \) is a relation on \( S \).
  - \( sR_a s' \) iff \( a \) can be performed at \( s \) and result in \( s' \)
- \( V : \text{AtProp} \rightarrow \mathcal{P}(S) \) is a function mapping each \( p \in \text{AtProp} \) to a set of states in \( S \) (representing where \( p \) is true).
A path is a sequence of states $s = s_0, s_1, s_2, s_3, \ldots$, where for each $i$, there is a concurrent action $a$ such that

$$s_i R_a s_{i+1}.$$ 

The path can be finite or infinite.

- Let $\ell(s)$ be the length of $s$ (it would be $\infty$ if infinite).
- Let $s_{\geq n} = s_n, s_{n+1}, \ldots$ be the suffix of $s$. 
The set of temporal logic (TL) formulas is the smallest set such that

- Any atomic property \( p \in \text{AtProp} \) is a formula and \( \perp \) ("false") is a formula
- if \( \varphi \) is a formula, so is \( \neg \varphi \), \( X\varphi \) ("next time \( \varphi \)"), \( G\varphi \) ("always (globally) will be \( \varphi \)"), and \( F\varphi \) ("eventually \( \varphi \)") are formulas.
- if \( \varphi, \psi \) are formulas, then \( \varphi \land \psi \) are formulas.

Those components in red are those not already in propositional logic.
Define truth of a formula on a path \( s = s_0, \ldots \) (finite or infinite) of a labelled transition system by a relation \( \models \):

\[
\begin{align*}
\mathbf{s} \models p & \iff s_0 \in V(p) \\
& \quad \text{(The first state along the path has } p) \\
\mathbf{s} \models \bot & \iff \text{never} \\
\mathbf{s} \models \neg \varphi & \iff \mathbf{s} \not\models \varphi \\
\mathbf{s} \models \varphi \land \psi & \iff \mathbf{s} \models \varphi \text{ and } \mathbf{s} \models \psi \\
\mathbf{s} \models X \varphi & \iff \text{if } \ell(s) > 1, \text{ then } \mathbf{s}_{\geq 1} \models \varphi \\
& \quad \text{(The next state (if exists) satisfies } \varphi) \\
\mathbf{s} \models G \varphi & \iff \text{for all } 0 < n < \ell(s), \mathbf{s}_{\geq n} \models \varphi \\
& \quad \text{(} \varphi \text{ in all future states)} \\
\mathbf{s} \models F \varphi & \iff \text{for some } 0 < n < \ell(s), \mathbf{s}_{\geq n} \models \varphi \\
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Define truth of a formula on a path $s = s_0, \ldots$ (finite or infinite) of a labelled transition system by a relation $\models$:

- $s \models p \iff s_0 \in V(p)$
  (The first state along the path has $p$)
- $s \models \bot$ never
- $s \models \neg \varphi \iff s \not\models \varphi$
- $s \models \varphi \land \psi \iff s \models \varphi$ and $s \models \psi$
- $s \models X\varphi \iff$ if $\ell(s) > 1$, then $s_{\geq 1} \models \varphi$
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- $s \models G\varphi \iff$ for all $0 < n < \ell(s)$, $s_{\geq n} \models \varphi$
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Temporal Logic Semantics

Define truth of a formula on a path $s = s_0, \ldots$ (finite or infinite) of a labelled transition system by a relation $|=:

- $s |-= p \iff s_0 \in V(p) $ (The first state along the path has $p$)
- $s |-= \bot$ never
- $s |-= \neg \varphi \iff s \not|= \varphi$
- $s |-= \varphi \land \psi \iff s |-= \varphi$ and $s |-= \psi$
- $s |-= X\varphi \iff$ if $\ell(s) > 1$, then $s_{\geq 1} |-= \varphi$ (The next state (if exists) satisfies $\varphi$)
- $s |-= G\varphi \iff$ for all $0 < n < \ell(s)$, $s_{\geq n} |-= \varphi$ ($\varphi$ in all future states)
- $s |-= F\varphi \iff$ for some $0 < n < \ell(s)$, $s_{\geq n} |-= \varphi$ ($\varphi$ is some future state)

- Note that $(s_0) |-= X\bot$ holds (vacuously) true
- Note that $(s_0, s_1, s_2, s_3) |-= XX\varphi$ iff $(s_2, s_3) |-= \varphi$
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- Note that $(s_0) \models X \bot$ holds (vacuously) true
- Note that $(s_0, s_1, s_2, s_3) \models XX \varphi$ iff $(s_2, s_3) \models \varphi$
let $\text{AtProp} = \{ e_0, e_1, e_2, e_3, e_4 \}$, where $e_k$ is the property that $k$ is eating in the current state.

- existence of deadlock is expressed using

$$\text{deadlock} \equiv X \perp \lor FX \perp$$

either the current state is a dead-end, or there will be a dead-end sometime in the future

- fairness is expressed using

$$\text{fairness} \equiv \neg \text{deadlock} \land \bigwedge_{k=0}^{4} GF e_k$$

There is no deadlock, and for every agent, at every point in the future, that agent will be able to eat at some later point.
Some classical problems involving temporal logic

- **Model Checking**: Given a logic, formula, and “model” (labelled transition system), determine whether the formula is true in the model.

- **Compositional Reasoning**: Find processes for components (the individual philosophers) and compose them into a labelled transition system. Express using a logic both properties of the components and properties of the composed system.
Social Phenomena
Example

A room has either one of two urns:
- Majority White: \( \{W, W, B\} \)
- Majority Black: \( \{B, B, W\} \).

People line up to enter the room one-by-one to:
- Draw a ball (observe and replace)
- Write down a guess (for all to see) as to which urn it is

In forming a guess, agents take into account:
- The outcome (ball) of their draw
- The guesses (which urn) that came before

A Cascade develops if agents’ conclusions are dominated by guesses that came before
- False cascade: a cascade where agents’ conclusions do not match the real situation
- Correct cascade: a cascade where agents’ conclusions do match the real situation
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Goal

We want a logic (to reason about cascades and related examples) with

- **Probabilistic** components (which urn is more likely)
- **Epistemic** components (beliefs agents have about each other)
- **Common knowledge** (of the rules of the example)
- **Dynamic updates** (to model how agents’ views change)
The Probabilistic Logic of Communication and Change is the synthesis of variants of the following logics:

- **Logic of Communication and Change for Dynamics of Common Knowledge**
  

- **Dynamic Epistemic Probabilistic Logic for Dynamics of Probabilities**
  
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Examples

- \(0.6 \cdot P_a(p) + 0.2 \cdot P_a([b]p) \geq 0.5\)
- \([(a \cup b)^*]p\) It is common knowledge among \(a\) and \(b\) that \(p\).
- \([e][a]p\) After informational event \(e\), \(a\) would believe \(p\).

PLCC

Language of the Probabilistic Logic of Communication and Change:

\[
\begin{align*}
\phi &::= \text{true} \mid p \mid \neg \phi \mid \phi \land \phi' \mid [\pi]\phi \mid [e]\phi \mid t_a \geq \beta \\
t_a &::= \alpha \cdot P_a(\phi) \mid t_a + t'_a \\
\pi &::= a \mid \pi; \pi' \mid \pi \cup \pi' \mid \pi^* \mid ?\phi
\end{align*}
\]

\(\phi\) (formulas)
\(t_a\) (probability terms)
\(\pi\) (epistemic programs)

\(p \in At\) (atomic proposition) such as the urn has majority white
\(a \in Ag\) (agent)
\(\alpha, \beta \in \mathbb{Q}\) (rational numbers)
\(e \in E\) (epistemic update event) such as drawing from urn
Static sublanguage

PE-PDL

Language of Probabilistic Epistemic Propositional Dynamic Logic: Same as PLCC, but with dynamic formulas $[e] \varphi$ removed.
Models

Definition (Bayesian Kripke models)

\[ M = (S, \sim, \mu, V) \]

- **S** - (non-empty) set of states
- \( \sim \) - equivalence relations \( \sim_a \) on \( S \), for each agent \( a \)
- \( \mu \) - indexed probability functions \( \mu_a : S \to (S \to [0, 1]) \), with values denoted \( \mu^s_a(s') \), and satisfying:
  - State-determined probability (SDP)
    each agent knows her probability
    if \( s \sim_a t \), then \( \mu^s_a(s') = \mu^t_a(s') \) for all \( s' \in S \)
  - Consistency (CONS)
    consistency of probabilities with knowledge
    \( \mu^s_a(t) = 0 \) if \( s \not\sim_a t \)
  - Caution (CAUT) agents are cautious
    \( s \not\sim_a t \) if \( \mu^s_a(t) = 0 \)
  - Probability (PROB) \( \mu^s_a \) is a probability mass function.
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- \( V : At \to \mathcal{P}(S) \) a valuation function.
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Select semantics of PE-PDL

\[ M, s \models p \iff s \in V(p) \]
\[ M, s \models [\pi]\phi \iff M, t \models \phi \text{ whenever } sR_{\pi} t \]
\[ M, s \models \sum_{j=1}^{n} \alpha_j P_{a}(\phi_j) \geq \beta \iff \sum_{j=1}^{n} \alpha_j \cdot \mu^s_{a}(\phi_j) \geq \beta \]

where \( \mu^s_{a}(\phi_j) \) abbreviates \( \sum_{s' \in S, s' \models \phi_j} \mu^s_{a}(s') \),

and \( R_{\pi} \) is a binary relation given by

\[ sR_{a} t \iff s \sim_{a} t \]
\[ sR_{\pi_1 \cup \pi_2} t \iff s(R_{\pi_1} \cup R_{\pi_2}) t \]
\[ sR_{\pi_1; \pi_2} t \iff sR_{\pi_1; R_{\pi_2}} t \text{ (there is } w, \text{ such that } sR_{\pi_1} w \text{ and } wR_{\pi_2} t) \]
\[ sR_{\pi^*} t \iff s(R_{\pi}^*) t \text{ ((} R_{\pi}^* \text{ the reflexive transitive closure of } R_{\pi}) \]
\[ sR_{\pi ? \phi} t \iff s = t \text{ and } M, s \models \phi \]
Event model

**Definition (Event Models)**

\[ A = (E, \sim, \Phi, \text{pre}, \text{sub}) \text{ where:} \]

- \( E \) - the (finite non-empty) set of **epistemic update events**.
- \( \sim \) - equivalence relations \( \sim_a \) on \( E \) for each agent \( a \).
- \( \Phi \) - finite pairwise inconsistent set of **precondition** formulas
- \( \text{pre} \) - functions \( \text{pre}_a : \Phi \rightarrow (E \rightarrow [0, 1]) \) for each \( a \in Ag \), \( \text{pre}_a(\phi) \) is a **subjective occurrence probability** function \( (\sum_{e \in E} \text{pre}_a(\phi)(e) = 1) \)
  
  \[ \text{PRE} : E \rightarrow \mathcal{P}(\Phi) \quad \text{PRE} : e \mapsto \{ \phi \mid \text{pre}(\phi)(e) > 0 \} \]

\[ \text{pre}_a(e \mid s) = \begin{cases} 
\text{pre}_a(\phi)(e) & \phi \in \Phi, M, s \models \phi \\
0 & \text{there is no such } \phi
\end{cases} \]

- \( \text{sub} \) - a **substitution function** \( \text{sub}(e) : At \rightarrow \mathcal{L}_{\text{PLCC}} \) for each \( e \in E \) (mapping all but finitely many \( p \in At \) to themselves).
A = (E, ∼, Φ, pre, sub) where:

- **E** - the (finite non-empty) set of epistemic update events.
- **∼** - equivalence relations ∼ₐ on E for each agent a.
- **Φ** - finite pairwise inconsistent set of precondition formulas
- **pre** - functions preₐ : Φ → (E → [0, 1]) for each a ∈ Ag, preₐ(φ) is a subjective occurrence probability function (∑ₑ∈E preₐ(φ)(e) = 1)

PRE : E → P(Φ) PRE : e ↦ {φ | pre(φ)(e) > 0}

preₐ(e | s) = \begin{cases} 
preₐ(φ)(e) & φ ∈ Φ, M, s ⊨ φ \\
0 & \text{there is no such } φ 
\end{cases}

- **sub** - a substitution function sub(e) : At → L_{PLCC} for each e ∈ E (mapping all but finitely many p ∈ At to themselves).
Definition (Event Models)

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\[
\text{PRE} : E \rightarrow \mathcal{P}(\Phi) \quad \text{PRE} : e \mapsto \{ \phi \mid \text{pre}(\phi)(e) > 0 \}
\]

\[
\text{pre}_a(e \mid s) = \left\{ \begin{array}{ll}
\text{pre}_a(\phi)(e) & \phi \in \Phi, M, s \models \phi \\
0 & \text{there is no such } \phi
\end{array} \right.
\]

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**Event model**

**Definition (Event Models)**

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**PRE** : E → ℙ(Φ)  
PRE : e ↦ {φ | pre(φ)(e) > 0}

pre_a(e | s) = \[
\begin{cases} 
  \text{pre}_a(φ)(e) & \text{φ ∈ Φ, } M, s \models φ \\
  0 & \text{there is no such } φ
\end{cases}
\]

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Definition (Event Models)

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# Event model

## Definition (Event Models)

\[ A = (E, \sim, \Phi, \text{pre}, \text{sub}) \]

where:

- **E** - the (finite non-empty) set of **epistemic update events**.
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**Joshua Sack**  
**Logic and its Applications to the Sciences**
**Definition (Product Update)**

\[ M = (S, \sim, \mu, V) \]  
Bayesian Kripke model

\[ A = (E, \sim, \Phi, \text{pre}, \text{sub}) \]  
event model

\[ M \otimes A = (S \otimes E, \sim, \mu, V) \]  
update product

- \[ S \otimes E \overset{\text{def}}{=} \{(s, e) \mid s \in S, e \in E, (M, s) \models \bigvee \text{PRE}(e)\} \].

- \((s, e) \sim_{a} (s', e')\) iff \(s \sim_{a} s'\) and \(e \sim_{a} e'\).

- Let \( D \overset{\text{def}}{=} \sum_{(s', e')} \sim_{a}(w, g) (\mu_{a}^{w}(s') \cdot \text{pre}_{a}(e' \mid s')) \), and put:

\[
\mu_{a}^{(w, g)}(s, e) \overset{\text{def}}{=} \begin{cases} 
\frac{\mu_{a}^{w}(s) \cdot \text{pre}_{a}(e \mid s)}{D} & \text{if } (s, e) \sim_{a} (w, g) \\
0 & \text{otherwise}
\end{cases}
\]

(Note that \( D \neq 0 \) for \((w, g) \in S \otimes E\).)

- \( V(p) = \{(s, e) \mid M, s \models \text{sub}(e)(p)\} \)}
Definition (Product Update)

\[ M = (S, \sim, \mu, V) \quad \text{Bayesian Kripke model} \]

\[ A = (E, \sim, \Phi, \text{pre}, \text{sub}) \quad \text{event model} \]

\[ M \otimes A = (S \otimes E, \sim, \mu, V) \quad \text{update product} \]

- \( S \otimes E \overset{\text{def}}{=} \{(s, e) \mid s \in S, e \in E, (M, s) \models \bigvee \text{PRE}(e)\} \).

- \((s, e) \sim_a (s', e')\) iff \( s \sim_a s'\) and \( e \sim_a e'\).

- Let \( D \overset{\text{def}}{=} \sum_{(s', e') \sim_a (w, g)} (\mu_a^w(s') \cdot \text{pre}_a(e' \mid s')) \), and put:

\[
\mu_a^{(w, g)}(s, e) \overset{\text{def}}{=} \begin{cases} \frac{\mu_a^w(s) \cdot \text{pre}_a(e \mid s)}{D} & \text{if } (s, e) \sim_a (w, g) \\ 0 & \text{otherwise} \end{cases}
\]

(Note that \( D \neq 0 \) for \((w, g) \in S \otimes E\).)

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Joshua Sack  Logic and its Applications to the Sciences
Definition (Product Update)

\[ M = (S, \sim, \mu, V) \] Bayesian Kripke model

\[ A = (E, \sim, \Phi, \text{pre, sub}) \] event model

\[ M \otimes A = (S \otimes E, \sim, \mu, V) \] update product

1. \( S \otimes E \overset{\text{def}}{=} \{(s, e) \mid s \in S, e \in E, (M, s) \models \bigvee \text{PRE}(e)\} \).

2. \((s, e) \sim_a (s', e')\) iff \( s \sim_a s' \) and \( e \sim_a e' \).

3. Let \( D \overset{\text{def}}{=} \sum_{(s', e')} (\mu^w_a(s') \cdot \text{pre}_a(e' \mid s')) \), and put:

\[
\mu^w_a(s, e) \overset{\text{def}}{=} \begin{cases} 
\frac{\mu^w_a(s) \cdot \text{pre}_a(e \mid s)}{D} & \text{if } (s, e) \sim_a (w, g) \\
0 & \text{otherwise}
\end{cases}
\]

(Note that \( D \neq 0 \) for \((w, g) \in S \otimes E\).)

4. \( V(p) = \{(s, e) \mid M, s \models \text{sub}(e)(p)\} \)
Definition (Product Update)

\[ M = (S, \sim, \mu, V) \] Bayesian Kripke model

\[ A = (E, \sim, \Phi, \text{pre}, \text{sub}) \] event model

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- \( S \otimes E \overset{\text{def}}{=} \{(s, e) \mid s \in S, e \in E, (M, s) \models \bigvee \text{PRE}(e)\} \).

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  0 & \text{otherwise}
  \end{cases}
  \]

  (Note that \( D \neq 0 \) for \( (w, g) \in S \otimes E \).)

- \( V(p) = \{(s, e) \mid M, s \models \text{sub}(e)(p)\} \)
The semantics of PLCC is the same as for PE-PDL, but with the following extra clause:

\[ M, s \models [e] \phi \quad \text{iff} \quad M, s \models \bigvee \text{PRE}(e) \text{ then } M \times A, (s, e) \models \phi, \]

where \( e \) is an event in action model \( A \).
Urn example with formulas

Basic elements of the language

- \( Ag \stackrel{\text{def}}{=} \{1, \ldots, n\} \) (agents)
- \( At \stackrel{\text{def}}{=} \{MW, MB\} \cup \{DW_i, DB_i, W_i, B_i\}_{i \in Ag} \) (atoms)
- \( E \stackrel{\text{def}}{=} \{dw_i, db_i, w_i, b_i\}_{i \in Ag} \) (events)

Some useful formulas and sets:

\[ \chi_i \stackrel{\text{def}}{=} (MW \lor MB) \land \bigwedge_{j<i}(DW_j \lor DB_j) \land \bigwedge_{j<i}(W_j \lor B_j) \land \bigwedge_{p \in At \geq i} \neg p \]

The situation right before \( i \) draws

\[ \chi_i^D \stackrel{\text{def}}{=} (MW \lor MB) \land \bigwedge_{j\leq i}(DW_j \lor DB_j) \land \bigwedge_{j<i}(W_j \lor B_j) \land \neg(W_i \lor B_i) \land \bigwedge_{p \in At > i} \neg p \]

The situation right before \( i \) writes

\[ At_{\geq i} \stackrel{\text{def}}{=} \{DW_j, DB_j, W_j, B_j\}_{i \leq j \leq n} \]
\[ At_{> i} \stackrel{\text{def}}{=} \{DW_j, DB_j, W_j, B_j\}_{i < j \leq n} \]
Depiction of Event Model: updates of probabilities

\( \text{pre}(\psi_i^W)(dw_i) = \frac{2}{3} \quad \text{pre}(\psi_i^B)(dw_i) = \frac{1}{3} \)
\( \quad \text{pre}(\psi_i^W)(db_i) = \frac{1}{3} \quad \text{pre}(\psi_i^B)(db_i) = \frac{2}{3} \)

\( \text{pre}(\phi_i^W)(w_i) = 1 \quad \text{pre}(\phi_i^B)(w_i) = 0 \)
\( \quad \text{pre}(\phi_i^W)(b_i) = 0 \quad \text{pre}(\phi_i^B)(b_i) = 1 \)

\( \psi_i^W \overset{\text{def}}{=} MW \land \chi_i \)

The urn has majority white and it is right before agent \( i \) draws a ball.

\( \psi_i^B \overset{\text{def}}{=} MB \land \chi_i \)

\( \phi_i^W \overset{\text{def}}{=} P_i(MW) > P_i(MB) \lor (DW_i \land P_i(MW) = P_i(MB)) \land \chi_i^D \)
i either considers the majority white urn more likely or
i considers them equally likely but had just drawn white.

\( \phi_i^B \overset{\text{def}}{=} P_i(MB) > P_i(MW) \lor (DB_i \land P_i(MW) = P_i(MB)) \land \chi_i^D \)
Depiction of Event Model: updates of atoms

\[
\begin{align*}
\text{sub}(\text{dw}_i, p) &= \begin{cases} 
\chi_i & p = \text{DW}_i \\
\not\chi_i & p \not= \text{DW}_i
\end{cases} \\
\text{sub}(\text{db}_i, p) &= \begin{cases} 
\chi_i & p = \text{DB}_i \\
\not\chi_i & p \not= \text{DB}_i
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\chi_i & \overset{\text{def}}{=} (\text{MW} \lor \text{MB}) \land \bigwedge_{j<i}(\text{DW}_j \lor \text{DB}_j) \\
& \quad \land \bigwedge_{j<i}(\text{W}_j \lor \text{B}_j) \land \bigwedge_{p \in \text{At}_{\geq i}} \neg p \\
\text{The situation right before } i \text{ draws}
\end{align*}
\]

\[
\begin{align*}
\chi_i^D & \overset{\text{def}}{=} (\text{MW} \lor \text{MB}) \land \bigwedge_{j\leq i}(\text{DW}_j \lor \text{DB}_j) \land \bigwedge_{j<i}(\text{W}_j \lor \text{B}_j) \\
& \quad \land \neg(\text{W}_i \lor \text{B}_i) \land \bigwedge_{p \in \text{At}_{> i}} \neg p \\
\text{The situation right before } i \text{ writes}
\end{align*}
\]

\[
\begin{align*}
\text{At}_{\geq i} & \overset{\text{def}}{=} \{ \text{DW}_j, \text{DB}_j, \text{W}_j, \text{B}_j \} \quad \text{if } i \leq j \leq n \\
\text{At}_{> i} & \overset{\text{def}}{=} \{ \text{DW}_j, \text{DB}_j, \text{W}_j, \text{B}_j \} \quad \text{if } i < j \leq n
\end{align*}
\]
\( \chi \overset{\text{def}}{=} (\text{MW} \lor \text{MB}) \land \neg(\text{MW} \land \text{MB}) \land \)

Exactly one urn is placed in the room

\( \land_{i \in \text{Ag}} (P_i(\text{MW}) = P_i(\text{MB})) \land \)

Each agent considers each urn equally likely

\( \land_{p \in \text{At}_{\geq 1}} \neg p \)

No one has drawn or written anything

\[ \left[ \bigcup_{i \in \text{Ag}} i \right]^* \chi \Rightarrow \]

\[ [\text{dw}_1][g_1][\text{dw}_2][g_2][f_3][g_3] \ldots [f_i][g_i] (P_k(\text{MW}) > P_k(\text{MB})) \]

For all \( 1 \leq j \leq i \), let \( f_j \in \{\text{dw}_j, \text{db}_j\} \) and \( g_j \in \{w_j, b_j\} \). Then

If it is common knowledge of \( \chi \), and the first two balls to be drawn are white, then after subsequent draws, every agent will consider the majority white urn to be more likely.
**Result**

$$\chi \overset{\text{def}}{=} (\text{MW} \lor \text{MB}) \land \neg(\text{MW} \land \text{MB}) \land$$

Exactly one urn is placed in the room

$$\land_{i \in \text{Ag}} (P_i(\text{MW}) = P_i(\text{MB})) \land$$

Each agent considers each urn equally likely

$$\land_{p \in \text{At}_1} \neg p$$

No one has drawn or written anything

---

**Proposition**

*For all $1 \leq j \leq i$, let $f_j \in \{\text{dw}_j, \text{db}_j\}$ and $g_j \in \{\text{w}_j, \text{b}_j\}$. Then*

$$[(\bigcup_{i \in \text{Ag}} i)^*] \chi \Rightarrow [\text{dw}_1][g_1][\text{dw}_2][g_2][f_3][g_3] \ldots [f_i][g_i](P_k(\text{MW}) > P_k(\text{MB}))$$

If it is common knowledge of $\chi$, and the first two balls to be drawn are white, then after subsequent draws, every agent will consider the majority white urn to be more likely.
Let

- $\text{MW}$ be a state whose only atom is $\text{MW}$ and
- $\text{MB}$ be a state whose only atom is $\text{MB}$

Any model satisfying $[(\bigcup_{k \in Ag} k)^*] \chi$ and $\text{MB}$ is bisimilar to:

\[
\begin{align*}
\text{all } k & : 1/2 & \text{all } k & : 1/2 \\
\text{MW} & \quad \quad & \text{MB} & \\
\end{align*}
\]

Recall

\[
\chi \overset{\text{def}}{=} (\text{MW} \lor \text{MB}) \land \neg (\text{MW} \land \text{MB}) \land \\
\text{Exactly one urn is placed in the room} \\
\bigwedge_{k \in Ag} (P_k(\text{MW}) = P_k(\text{MB})) \land \\
\text{Each agent considers each urn equally likely} \\
\bigwedge_{p \in At_{\geq 1}} \neg p \\
\text{No one has drawn or written anything}
\]
Agent 1 considers the majority white urn to be more likely, and will write white.
After agent 1 draws white

Agent 1 considers the majority white urn to be more likely, and will write white.
The result of agent 1’s guess is bisimilar (via generated submodel) to the following:
Agent 2 considers the majority white urn to be more likely, and will write white.
After second agent draws white

Agent 2 considers the majority white urn to be more likely, and will write white.
After agent 2 writes

The result of agent 2’s guess is bisimilar (via generated submodel) to the following:

\[
\begin{align*}
\text{MW} & : dw_1w_1 \\
& dw_2w_2
\end{align*}
\]

\[
\begin{align*}
\text{MB} & : dw_1w_1 \\
& dw_2w_2
\end{align*}
\]

\[
k : 4/5
\]

\[
\text{all } k
\]

\[
k : 1/5
\]
Agent 3 considers the majority white urn to be more likely, and will write white.
Agent 3 considers the majority white urn to be more likely, and will write white.
Result of agent 3 writing

\[ \forall k \neq 3 \quad \models P_k(MW) = \frac{12}{15} \wedge P_k(MB) = \frac{3}{15}. \]
\( (\models P_k(MW) = \frac{4}{5} \wedge P_k(MB) = \frac{1}{5}.) \)
This is similar to the situation after agent 2 wrote.
\[ P_k(MW) = \frac{12}{15} \land P_k(MB) = \frac{3}{15}. \]

This is similar to the situation after agent 2 wrote.

Joshua Sack  Logic and its Applications to the Sciences
Higher order reasoning (even common knowledge) cannot necessarily prevent false cascades.
Logic can be used to help reason about very different subjects matters, ranging from physics, computer science, to social science.

- In quantum physics, logic can help us understand conditions that require quantum contextuality, making it easier to find experiments that support contextually.
- In computation and concurrency, logic can help us reason about problems whether a program will “crash” or whether a concurrent system is “fair”.
- In social science, logic can be used to reason formally about complex group behaviors, such as informational cascades.
Thank you!