Refined inversion statistics on permutations

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Definition

A permutation of rank \( n \) is a bijective function
\[ \pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}. \]

Definition

An inversion in a permutation \( \pi \) is a pair \((a, b)\), such that
\[ 0 \leq a < b \leq n \text{ and } \pi_a > \pi_b. \]
A non-inversion is a pair \((a, b)\), such that 0 \leq a < b \leq n and
\[ \pi_a < \pi_b. \]

Example

\( \pi = 32415 \) has
- four inversions: \((1, 2), (1, 4), (2, 4), (3, 4)\)
- six non-inversions: \((1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\)
Non-inversion Sums

INV(π) is the set of all inversions of π and NINV(π) is the set of all non-inversions of π.

**Definition**

- The **inversion sum** of π is given by
  \[
  \text{invsum}(\pi) = \sum_{(a,b) \in \text{INV}(\pi)} (b - a)
  \]

- The **non-inversion sum** of π is given by
  \[
  \text{ninvsum}(\pi) = \sum_{(a,b) \in \text{NINV}(\pi)} (b - a)
  \]

**Proposition**

\[
\sum_{(a,b) \in \text{NINV}(\pi)} \pi(b) - \pi(a) = \sum_{(a,b) \in \text{NINV}(\pi)} b - a
\]
Cosine of a permutation

Let $1$ be the identity permutation of rank $n$.

**Definition**

For any permutation $\pi$ of rank $n$, the cosine of $\pi$ is

$$\cos(\pi) = 1 \cdot \pi = \sum_{i=1}^{n} i \pi(i)$$

**Observation**

Given a permutation $\pi$ of rank $n$, if $\theta$ is the angle between the vectors corresponding to $\pi$ and $1$.

$$\cos(\pi) = a(n) \cos(\theta),$$

where $a(n) = n(n + 1)(2n + 1)/6$. 

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For \( k \geq 35 \), there exists a permutation \( \pi \) such that
\[
\cos(\pi) = 1 \cdot \pi = k
\]
The total number of permutations \( \pi \) such that \( \cos(\pi) = k \) is given by the sequence

A135298 in the *Online Encyclopedia of Integer Sequences*.

Our theorem shows that this sequence is non-zero after \( k = 34 \).
Non-inversion sum and cosine

If $\pi = \pi_1 \cdots \pi_n$, then $\pi^c = (n + 1 - \pi_1) \cdots (n + 1 - \pi_n)$.

**Theorem**

\[
\cos(\pi) = 1 \cdot 1^c + \text{ninvsum}(\pi) = \binom{n+2}{3} + \text{ninvsum}(\pi)
\]

**Theorem**

*For* $n \geq 4$ *and* $0 \leq k \leq \binom{n+1}{3}$, *there is a permutation* $\pi$ *such that* $\text{ninvsum}(\pi) = k$.

**Theorem**

*For* $n \geq 6$,*

\[
\binom{n+1}{3} + \binom{n}{3} \geq \binom{n+2}{3} - 1
\]
Given a permutation $\pi$ of rank $n$, its non-inversion zone-crossing vector is $\text{nzcv}(\pi) = (z_1, z_2, \ldots, z_n)$, where $z_k$ is the number of non-inversions $(a, b) \in \text{NINV}(\pi)$, where $a \leq k < b$.

The sum of the coordinates of $\text{nzcv}(\pi)$ is equal to $\text{ninvsum}(\pi)$. 

**Definition**

**Proposition**
Let $\text{nzcv}(\pi)_k$ be the $k^{th}$ coordinate of $\text{nzcv}(\pi)$.

**Theorem**

The number of permutations of rank $n$, such that $\text{nzcv}(\pi)_k = j$ is

$$k!(n - k)! \left[ q^j \right] \left[ \begin{array}{c} n \\ k \end{array} \right]_q,$$

where for any polynomial $p(q)$, its coefficient of $q^j$ is denoted by $\left[ q^j \right] p(q)$, and

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [n]_q! = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q}.$$
We are interested in the distribution function for non-inversion sums:

\[ N_n(q) = \sum_{\pi \in S_n} q^{n\text{invsum}(\pi)} \]

**Theorem**

\[ N_{n+1}(q) = N_n(q) + \sum_{k=1}^{n-1} q^{\binom{k+1}{2}} \sum_{\pi \in S_n} q^{n\text{ncv}(\pi)_k} q^{n\text{invsum}(\pi)} + q^{\binom{n+1}{2}} N_n(q). \]
THANK YOU!