CHARACTERIZING $C(X)$ AMONG INTERMEDIATE C-RINGS ON $X$

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Abstract. Let $X$ be a completely regular topological space. An intermediate ring is a ring $A(X)$ of continuous functions satisfying $C^*(X) \subseteq A(X) \subseteq C(X)$. We give a characterization of $C(X)$ in terms of extensions of functions in $A(X)$ to real-compactifications of $X$. We also give equivalences of properties involving the closure in the real-compactifications of $X$ of zero-sets in $X$; we use these equivalences to answer an open question about the correspondences of ideals in intermediate rings and $z$-filters on $X$.

1. Introduction

The ring $C(X)$ of all continuous real-valued functions on the completely regular Hausdorff space $X$ has been characterized among its subrings in various ways. Perhaps the best known characterization is the Stone-Weierstrass Theorem: If $X$ is compact, then any subring of $C(X)$ which contains the constants, separates points, and is uniformly closed is $C(X)$ itself [16, p. 291]. Other characterizations can be found in the book Characterizations of $C(X)$ among its Subalgebras by R. B. Burkel [2]. In this article we consider intermediate subrings of $C(X)$, that is, subrings $A(X)$ satisfying

$$C^*(X) \subseteq A(X) \subseteq C(X)$$
where $C^*(X)$ is the subring of $C(X)$ consisting of the bounded functions. In [11], to each $A(X)$ there is a realcompact space $v_A X$ to which each $f \in A(X)$ can be continuously extended to a function $f^{\text{v} A}$ defined on all of $v_A X$, and $f^\ast$ can be further extended to all of $\beta X$ via the Stone extension of $f$, denoted $f^\ast$. A $C$-ring is a ring $A(X)$ that is isomorphic to $C(Y)$ for some completely regular space $Y$. In this article we give a characterization of $C(X)$ among intermediate $C$-rings $A(X)$ in terms of the extension of functions to $v_A X$ (Theorem 3.7), and we use this characterization to give a more direct proof of a result by Swardson [14, Corollary 4.3]. We also characterize intermediate $C$-rings that have the property that the closure of a zero-set in $v_A X$ is a zero-set (Theorem 4.2). We furthermore characterize $C^*(X)$ among intermediate rings $A(X)$ in terms of the relationship between the extension $f^{\text{v} A}$ and the Stone extension $f^\ast$ of $f$ for each $f$ (Theorem 3.10). In the setting of this paper, we give a more direct proof of a result by Rudd (the equivalence of the first two parts of [11, Theorem 1.2]) concerning the relationship between $f$ and $f^\ast$ (Theorem 3.11 in this paper). Furthermore, we give a class of intermediate rings for which the natural correspondence between ideals and $z$-filters maps maximal ideals to $z$-ultrafilters (Theorem 5.2). This answers [13] Question 4 in the negative: it is not the case that $A(X) = C(X)$ if and only if every maximal ideal in $A(X)$ is mapped by this correspondence to a $z$-ultrafilter on $X$.

2. Preliminaries

We follow the definitions and notation of [12, 13]. Let $X$ be a completely regular space. A zero-set in $X$ is a set of the form $Z(f) = \{ x \in X : f(x) = 0 \}$ for some $f$ in $A(X)$; $Z[X]$ denotes the family of all zero-sets of $X$. It is well known (see [7]) that $Z$ acts as a correspondence between proper ideals $I$ in $C(X)$ and $z$-filters $Z[I] = \{ Z(f) : f \in I \}$ on $X$. There is a different but well known mapping $E$ mapping ideals in $C^*(X)$ to $z$-filters on $X$, given by $E[I] = \bigcup \{ E_\epsilon(f) : f \in I, \epsilon > 0 \}$, where $E_\epsilon(f) = \{ x : |f(x)| \leq \epsilon \}$ (see [7]). However, $E$ and $Z$ fail to be correspondences for intermediate rings $A(X)$ strictly between $C^*(X)$ and $C(X)$ and $z$-filters on $X$ [13, Propositions 3.2 and 4.2]. Thus extensions $Z_A$ of $E$ and $3_A$ of $Z$ to intermediate rings were defined as follows. If $f \in C(X)$ and $E \subseteq X$, we say that $f$ is $E$-regular in $A(X)$ if there exists $g \in A(X)$ such that $f(x) \cdot g(x) = 1$ for all $x \in E$. We may simply write that $f$ is $E$-regular if $A(X)$ is understood from context. For $f \in A(X)$ we have

$$Z_A(f) = \{ E \in Z[X] : f \text{ is } E^\circ \text{-regular} \}$$

$$3_A(f) = \{ E \in Z[X] : f \text{ is } H \text{-regular for all zero-sets } H \subseteq E^\circ \}.$$
Note the difference between $\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A(f)$. For example, if $A(X) = C(X)$ then $\mathfrak{Z}_A(f) = (\mathcal{Z}(f))$, the $z$-filter consisting of all zero-sets containing $\mathcal{Z}(f)$, whereas $\mathcal{Z}_A(f)$ consists of all zero-set neighborhoods of $\mathcal{Z}(f)$. For each non-invertible $f$ in $A(X)$, both $\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A(f)$ are $z$-filters on $X$ and $\mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f)$.

For an ideal $I$ in $A(X)$ we define $Z_A[I] = \bigcup \{ Z_A(f) : f \in I \}$ and $\mathfrak{Z}_A[I] = \bigcup \{ \mathfrak{Z}_A(f) : f \in I \}$. Again $Z_A[I]$ and $\mathfrak{Z}_A[I]$ are $z$-filters on $X$. The maps $Z_A$ and $\mathfrak{Z}_A$ between ideals of $A(X)$ and $z$-filters on $X$ do extend the correspondences $E$ and $Z$ respectively to $A(X)$. In particular, for $A(X) = C^*(X)$ we have $Z_A[I] = E[I]$ (from [8, Corollary 1.3]), and for $A(X) = C(X)$ we have $\mathfrak{Z}_A[I] = Z[I]$ (from [8, Corollary 2.4]).

The hull of a $z$-filter $F$ is the set

$$hF = \{ U \mid U \text{ is a } z\text{-ultrafilter and } F \subseteq U \}. $$

Also, the kernel of a set $\mathcal{U}$ of $z$-ultrafilters is

$$k\mathcal{U} = \bigcap_{U \in \mathcal{U}} U$$

(as the intersection of $z$-filters is a $z$-filter, $k\mathcal{U}$ is a $z$-filter). From [8, Theorem 3.1] we have the following relationship between $Z_A$ and $\mathfrak{Z}_A$:

$$\mathfrak{Z}_A(f) = khZ_A(f)$$

for each $f \in A(X)$. The following is from [9].

**Lemma 2.1.** Let $A(X)$ be an intermediate ring, let $f \in A(X)$, and let $F$ be a $z$-filter on $X$. Then $Z_A(f) \subseteq F$ if and only if $\lim_F fh = 0$ for all $h \in A(X)$.

For a $z$-filter $F$ on $X$ we define the inverse maps $Z_A^{-} [F] = \{ f \in A(X) : Z_A(f) \subseteq F \}$ and $\mathfrak{Z}_A^{-} [F] = \{ f \in A(X) : \mathfrak{Z}_A(f) \subseteq F \}$. It is shown in [9] that for any intermediate ring $A(X)$, $Z_A^{-}$ maps $z$-filters on $X$ to ideals in $A(X)$, and it is shown in [12, Theorem 14] that the corresponding result for $\mathfrak{Z}_A^{-}$ holds for intermediate $C$-rings $A(X)$. From the definition of $Z_A^{-}$ and Lemma 2.1 we obtain the following.

**Lemma 2.2.** Let $A(X)$ be an intermediate ring and let $F$ be a $z$-filter on $X$. Then $Z_A^{-} [F] = \{ f \in A(X) : \lim_F fh = 0 \text{ for all } h \in A(X) \}$. 

In general, if $M$ is a maximal ideal then $\mathfrak{Z}_A[M]$ and $Z_A[M]$ need not be $z$-ultrafilters, but each is contained in a unique $z$-ultrafilter (see [3] and [8] respectively). However, if $U$ is a $z$-ultrafilter on $X$, then $Z_A^{-} [U]$ and $\mathfrak{Z}_A^{-} [U]$ are maximal ideals in $A(X)$, as the following lemma from [8] describes.

**Lemma 2.3.** Let $A(X)$ be an intermediate ring, and let $U$ be a $z$-ultrafilter on $X$. Then the following hold:
(a) Each of $Z_A^\rightarrow[\mathcal{U}]$ and $Z_A^\leftarrow[\mathcal{U}]$ are maximal ideals in $A(X)$ and $Z_A^\rightarrow[\mathcal{U}] = 3_A^\rightarrow[\mathcal{U}]$.

(b) The map $\mathcal{U} \to 3_A^\leftarrow[\mathcal{U}]$ defines a one-one correspondence between $z$-ultrafilters on $X$ and maximal ideals in $A(X)$.

Recall that the Stone-Čech compactification $\beta X$ of $X$ can be realized as the set of $z$-ultrafilters on $X$ topologized with the hull-kernel topology, and $X$ is naturally embedded in $\beta X$ by the map $p \mapsto U_p$, where $U_p = \{\{p\}\}$. So a point $p$ in $\beta X$ corresponds to the $z$-ultrafilter $U_p$ on $X$. Throughout this paper, we identify each $p \in \beta X$ with $U_p$. Then the closure of any set $U \subseteq \beta X$ is $hk(U)$, the hull of the kernel of $U$. By the preceding result $Z_A^\rightarrow[U_p]$ is a maximal ideal which we denote by $M_A^p$. Thus the correspondence $p \mapsto M_A^p$ is one-one between $\beta X$ and the maximal ideals of $A(X)$.

A $z$-ultrafilter $\mathcal{U}$ on $X$ is called $A$-stable if for every $f$ in $A(X)$ there is a member of $\mathcal{U}$ on which $f$ is bounded. For each $A(X)$ we define a subspace $\nu A X$ of $\beta X$, called the $A$-compactification of $X$, by $\nu A X = \{p \in \beta X : U_p$ is $A$-stable\} [4]. If $A(X) = C^*(X)$ then $\nu A X = \beta X$ and if $A(X) = C(X)$ then $\nu A X = \nu X$, the Hewitt realcompactification of $X$. We have $X \subseteq \nu X \subseteq \nu A X \subseteq \beta X$.

It is shown in [4] that the space $\nu A X$ consists of the points of $\beta X$ to which every $f$ in $A(X)$ can be continuously extended. We denote the extension of $f$ to $\nu A X$ by $f^{\nu A}$. It is also shown in [4] that for $f \in A(X)$ and $p \in \nu A X$ we have

\begin{equation}
(2.2) \quad f^{\nu A}(p) = \lim_{U_p} f.
\end{equation}

**Theorem 2.4** (from [4]). An intermediate ring $A(X)$ is a $C$-ring if and only if $A(X)$ is isomorphic to $C(\nu A X)$, via the map $f \mapsto f^{\nu A}$.

Different intermediate rings may induce the same realcompact extension of $X$. For additional information on the realcompactifications of $X$, see [1, 4, 6].

3. **The Zero-Sets of $f$, $f^{\nu A}$, and $f^*$**

If $A(X)$ is an intermediate $C$-ring, $f \in A(X)$, and $f^*$ is the Stone extension of $f$ (to be defined below), we show the following containments:

$$cl_{\nu A X} Z(f) \subseteq Z(f^{\nu A}) \subseteq Z(f^*)$$

The first containment, that $cl_{\nu A X} Z(f) \subseteq Z(f^{\nu A})$, is Proposition 3.2 and this containment can be strict by Example 3.3. Furthermore, equality of
Lemma 3.1. Let $A(X)$ be an intermediate ring and $f \in A(X)$. Then
\[ Z(f) = \bigcap \{ E : E \in Z_A(f) \} = \bigcap \{ E : E \in 3_A(f) \}. \]

Proof. It is proved in [10, Proposition 2.2] that $Z(f) = \bigcap \{ E : E \in Z_A(f) \}$, so it is sufficient to show that $\bigcap 3_A(f) = \bigcap Z_A(f)$. Since by (2.1), $Z_A(f) \subseteq 3_A(f)$, it follows that $\bigcap 3_A(f) \subseteq \bigcap Z_A(f)$.

For the other containment, suppose that $p$ is a point of $X$ such that $p \in \bigcap Z_A(f)$; then $u_p \supseteq Z_A(f)$. Since by (2.1), $3_A(f) = khZ_A(f)$, we have that $3_A(f) \subseteq u_p$, and it follows that $p \in \bigcap u_p \subseteq \bigcap 3_A(f)$. \qed

The $A$-stable hull of a $z$-filter $F$ is defined as in [12]:
\[ h^A(F) = \{ u_p : u_p \text{ is an } A\text{-stable } z\text{-ultrafilter on } X, \text{ and } F \subseteq u_p \}. \]

The closure operator in the topology of $v_A X$ is $h^A k$, because $v_A X$ is a subspace of $\beta X$ (where the closure operator is $hk$). If $E$ is a zero-set in $X$, then $kE = \langle E \rangle$, where $\langle E \rangle$ denotes the $z$-filter consisting of all zero-sets containing $E$. So
\[ (3.1) \quad cl_{v_A X} E = h^A k E = h^A(E). \]

In other words $cl_{v_A X} E$ consists of the $A$-stable $z$-ultrafilters that contain $E$.

Proposition 3.2. If $A(X)$ is an intermediate $C$-ring and $f \in A(X)$, then
\[ cl_{v_A X} Z(f) \subseteq Z(f^{\nu A}). \]

Proof. Since $Z(f)$ is a zero-set in $X$, it immediately follows from (3.1) that $cl_{v_A X} Z(f) = h^A(Z(f))$. By Lemma 3.1, $3_A(f) \subseteq (Z(f))$, so $cl_{v_A X} Z(f) = h^A((Z(f))) \subseteq h^A 3_A(f)$. Also $3_A(f) = khZ_A(f)$, and by [12] Lemma 5, $khZ_A(f) = kh^A Z_A(f)$. So $h^A 3_A(f) = h^A khZ_A(f) = h^A kh^A Z_A(f) = cl_{v_A X} h^A Z_A(f)$. It is shown in [12] Lemma 4 that $h^A Z_A(f) = Z(f^{\nu A})$; so we have $cl_{v_A X} h^A Z_A(f) = cl_{v_A X} Z(f^{\nu A})$. But $Z(f^{\nu A})$ is a closed set in $v_A X$, because it is a zero-set. So $cl_{v_A X} Z(f^{\nu A}) = Z(f^{\nu A})$. This completes the proof. \qed

The following example shows that in general the containment in Proposition 3.2 may be strict.

Example 3.3. Let $X = \mathbb{N}$ and $A(X) = C^*(\mathbb{N})$. Suppose $E$ is the set of even numbers in $\mathbb{N}$, and let $g$ be the functions whose value is zero on...
the set $E$ and $1/n$ at the odd integer $n$. Then $cl_{\beta X} Z(g)$ consists of all $z$-ultrafilters that contain the set $E$ of even numbers in $\mathbb{N}$, and hence none of these $z$-ultrafilters contain the set of odd numbers in $\mathbb{N}$. Now for any free $z$-ultrafilter $U_p$ containing the set of odd integers we have $\lim_{U_p} g = 0$. So $p \in Z(g^0)$, but clearly $p \notin cl_{\beta X} Z(g)$. Thus $cl_{\beta X} Z(g) \subseteq Z(g^0)$ (strict containment).

Note that this argument does not work for every function with zero-set $E$. Consider $h$ which maps $x$ to 1 if $x \notin E$ and to 0 otherwise; that is, $h$ is the characteristic function of the complement of $E$. Here $cl_{\beta X} Z(h)$ consists of all $z$-ultrafilters that contain $E$. As $\mathbb{N}$ is discrete, any $z$-ultrafilter $U_p$ not in $cl_{\beta X} Z(h)$ cannot contain $E$, and hence must contain the complement of $E$. But then $\lim_{U_p} h = 1$ for such a $z$-ultrafilter $U$, and hence by (2.2), $U_p \notin Z(h^0)$. Thus $cl_{\beta X} Z(h) = Z(h^0)$.

As we will see in the next lemma, there are a number of characterizations of when $cl_{\nu X} Z(f)$ is equal to $Z(f^\omega)$. One of them is the negation of Condition ii of a theorem of Rudd [11, Theorem 1.2] that is expressed in terms of near-zero-sets. Rudd defines a subset $T$ of $X$ to be a near-zero-set of $f \in C(X)$ if for every $\epsilon > 0$ there exists a point $p \in T$ such that $|f(p)| < \epsilon$. The significance of $T$ being a “near-zero-set” of $f$ is that $f$ is not $T$-regular with respect to $C^\ast(X)$. So in the context of intermediate rings we make the following definition.

**Definition 3.4.** Let $A(X)$ be an intermediate ring of continuous functions and let $f \in C(X)$. A subset $T$ of $X$ is called a near-zero-set of $f$ with respect to $A(X)$ if $f$ is not $T$-regular in $A(X)$.

It is apparent that a set $E$ is a near-zero-set of $f$ in the sense of Rudd [11] if and only if $E$ is a near-zero-set of $f$ with respect $C^\ast(X)$ in the sense of Definition 3.4. Also, if $B(X) \subseteq A(X)$ and if $E$ is a near-zero-set of $f$ with respect to $A(X)$, then $E$ is a near-zero-set of $f$ with respect to $B(X)$.

**Lemma 3.5.** Let $A(X)$ be an intermediate ring. Then for each $f \in A(X)$, the following are equivalent:

(a) $\mathfrak{z}_A(f) = \langle Z(f) \rangle$.

(b) No near-zero-set of $f$ with respect to $A(X)$ is completely separated from $Z(f)$.

**Proof.** [b] $\Rightarrow$ [a] Suppose that $\mathfrak{z}_A(f) \neq \langle Z(f) \rangle$. Then by Lemma 3.1 there exists a zero-set $E$ such that $Z(f) \subseteq E$ and $E \notin \mathfrak{z}_A(f)$. By the definition of $\mathfrak{z}_A(f)$ it follows that there exists a zero-set $H$ in the complement of $E$ for which $f$ is not $H$-regular with respect to $A(X)$. So $H$ is a near-zero-set of $f$ with respect to $A(X)$. But $Z(f)$ and $H$ are disjoint zero-sets in $X$ and hence are completely separated ([11], page 17).
Suppose that there exists a near-zero-set $T$ of $f$ with respect to $A(X)$ which is completely separated from $Z(f)$. So there is a zero-set $H$ disjoint from $Z(f)$ such that $T \subseteq H$. Since $f$ is not $T$-regular, it follows that $f$ is not $H$-regular either. So $H$ is a zero-set in the complement of $Z(f)$ and $f$ is not $H$-regular in $A(X)$. It follows from the definition of $3_A(f)$ that $Z(f) \not\subseteq 3_A(f)$, and so $3_A(f) \neq \langle Z(f) \rangle$.

In the case of $C$-rings, we have the following lemma.

**Lemma 3.6.** Let $A(X)$ be an intermediate $C$-ring. Then for each $f \in A(X)$, the following are equivalent:

(a) $\text{cl}_{v_A X} Z(f) = Z(f^{v_A})$.
(b) $3_A(f) = \langle Z(f) \rangle$.

**Proof.** (a) $\Rightarrow$ (b) Suppose $\text{cl}_{v_A X} Z(f) = Z(f^{v_A})$. By Lemma 3.1, $3_A(f) \subseteq \langle Z(f) \rangle$. For the other inclusion, suppose $E$ is a zero-set such that $E \supseteq Z(f)$. Then $\text{cl}_{v_A X} E \supseteq \text{cl}_{v_A X} Z(f) = Z(f^{v_A})$ by hypothesis. But by [12, Theorem 6], $3_A(f) = \{ E \in Z(X) \mid Z(f^{v_A}) \subseteq \text{cl}_{v_A} E \}$. Hence we have $E \in 3_A(f)$.

(b) $\Rightarrow$ (a) Suppose that $3_A(f) = \langle Z(f) \rangle$. Then

\[
\text{cl}_{\beta X} Z(f^{v_A}) = \text{cl}_{\beta X} h^A Z_A(f) \quad \text{(by [12, Lemma 4])}
= hh^A Z_A(f) \quad \text{(by definition of closure in } \beta X\text{)}
= hh Z_A(f) \quad \text{(by [12, Lemma 5])}
= h3_A(f) \quad \text{(by (2.1))}
= h\langle Z(f) \rangle \quad \text{(by hypothesis)}
\]

Now, since $Z(f^{v_A})$ is closed in $v_A X$, we have

\[
Z(f^{v_A}) = \text{cl}_{v_A X} Z(f^{v_A}) = \text{cl}_{\beta X} Z(f^{v_A}) \cap v_A(X)
= h\langle Z(f) \rangle \cap v_A(X) = h^A Z(f) = \text{cl}_{v_A X} Z(f).
\]

As a special case of Proposition 3.2, we have that $\text{cl}_{\beta X} Z(f) \subseteq Z(f^\beta)$ and we know from Example 3.3 that the containment can be strict. But in general $\text{cl}_{v_X} Z(f) = Z(f^v)$, where $vX$ is the Hewitt realcompactification of $X$ (see [7, §8.8(b)]). The next theorem shows that this latter property characterizes $C(X)$ among intermediate $C$-rings on $X$.

**Theorem 3.7.** Let $A(X)$ be an intermediate $C$-ring. Then $A(X) = C(X)$ if and only if every $f \in A(X)$ satisfies $\text{cl}_{v_A X} Z(f) = Z(f^{v_A})$. 
It follows by Lemma 2.1 that a

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Proposition 3.9.

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Proof.

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Corollary 3.8.

[14, Corollary 4.3]

satisfies

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We observe that for any intermediate ring

hand, by [12, Lemma 4] we have

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By (3.2), the zero-set of

Proof.

The desired result is immediate from the following equivalences:

\[ A(X) = C(X) \]

\[ \iff \exists A(f) = \langle Z(f) \rangle \text{ for each } f \in A(X) \quad \text{(by [8, Theorem 2.3])} \]

\[ \iff cl_{v_AX} Z(f) = Z(f^A) \text{ for each } f \in A(X) \quad \text{(by Lemma 3.6).} \quad \square \]

Swardson [14] Corollary 4.3 shows that a completely regular space \( X \) satisfies \( cl_{\beta X} Z(f) = Z(f^\beta) \) for every \( f \in C^*(X) \) if and only if \( X \) is pseudocompact. We show that this characterization follows from Theorem 3.7 above. First, recall that a space \( X \) is pseudocompact if every continuous real-valued function on \( X \) is bounded. So \( X \) is pseudocompact if and only if \( C(X) = C^*(X) \).

**Corollary 3.8.** [14] Corollary 4.3 Let \( X \) be a completely regular space. Then \( cl_{\beta X} Z(f) = Z(f^\beta) \) for every \( f \in C^*(X) \) if and only if \( X \) is pseudocompact.

Proof. Suppose that \( cl_{\beta X} Z(f) = Z(f^\beta) \) for every \( f \in C^*(X) \). Setting \( A(X) = C^*(X) \), we have that \( v_A X = \beta X \), and hence \( cl_{v_A X} Z(f) = Z(f^A) \), for every \( f \in A(X) \). It follow by Theorem 3.5 that \( A(X) = C(X) \). But this means that \( C^*(X) = C(X) \), and so \( X \) is pseudocompact.

Conversely, suppose that \( X \) is pseudocompact and set \( A(X) = C^*(X) \). As \( C^*(X) = C(X) \), we also have \( A(X) = C(X) \), and it follows by Theorem 3.7 that \( cl_{v_A X} Z(f) = Z(f^A) \). But since \( A(X) = C^*(X) \) we have that \( v_A X = \beta X \), and it follows that \( cl_{\beta X} Z(f) = Z(f^\beta) \) for every \( f \in C^*(X) \).

\[ \square \]

For \( f \in C(X) \), the function \( f^* \) denotes the Stone extension of \( f \) to the one-point compactification \( \mathbb{R}^* = \mathbb{R} \cup \{\infty\} \). This means that \( f^* : \beta X \to \mathbb{R}^* \) extends the function \( f^A \), and that (2.2) extends as follows: every point \( p \in \beta X \) corresponds to a \( z \)-ultrafilter \( U_p \) on \( X \), and the value of \( f^* \) at \( p \) is

(3.2)

\[ f^*(p) = \lim_{U_p} f. \]

We observe that for any intermediate ring \( A(X) \), the zero-set of \( f^A \) is contained in the zero-set of \( f^* \), as follows.

**Proposition 3.9.** If \( A(X) \) is an intermediate ring and \( f \in A(X) \), then

\[ Z(f^A) \subseteq Z(f^*). \]

Proof. By (3.2), the zero-set of \( f^* \) is \( Z(f^*) = \{ p \in \beta X : \lim_{U_p} f = 0 \} \). It follows by Lemma 2.1 that a \( z \)-ultrafilter \( U_p \) that contains \( Z_A(f) \) must satisfy \( \lim_{U_p} f h = 0 \) for every \( h \in A(X) \), and in particular for \( h = 1 \). Hence \( U_p \) belongs to \( Z(f^*) \), that is \( hZ_A(f) \subseteq Z(f^*) \). On the other hand, by [12, Lemma 4] we have \( Z(f^A) = h^A Z_A(f) \). Thus \( Z(f^A) = h^A Z_A(f) \subseteq h Z_A(f) \subseteq Z(f^*). \quad \square \)
Theorem 3.10. Let $A(X)$ be an intermediate ring. Then $A(X) = C^*(X)$ if and only if every $f \in A(X)$ satisfies $Z(f^{\beta_X}) = Z(f^*)$.

Proof. If $A(X) = C^*(X)$ then $v_A X = \beta X$ and $f^{v_A} = f^\beta$. Also, as $f$ is bounded, $f^*$ never attains the value $\infty$, and hence $f^{v_A} = f^*$. So $Z(f^{v_A}) = Z(f^*)$ for every $f \in C^*(X)$.

Conversely, suppose that $A(X) \neq C^*(X)$. Then there is an unbounded function $g \in A(X)$. Without loss of generality assume that $g > 0$, and let $f = 1/g$. Let $U$ be any $z$-ultrafilter containing the zero-sets $\{ x \in X : g(x) \geq n \}$. Clearly $\lim_{U} f = 0$, so $U \in Z(f^*)$. But $U$ is not $A$-stable because $f$ is unbounded on every set in $U$, so $U \notin Z(f^{v_A})$. \qed

Rudd [11] considers the question of when $Z(f^*) = cl_{\beta X} Z(f)$. We give a short proof of the equivalence of the first two parts of [11, Theorem 1.2] by using the results of this article.

Theorem 3.11. Let $f \in C(X)$. Then $Z(f^*) \neq cl_{\beta X} Z(f)$ if and only if there is a near-zero-set of $f$ with respect to $C^*(X)$ which is completely separated from $Z(f)$.

Proof. First, note that two sets in $X$ are completely separated if and only if they are contained in disjoint zero-sets [7, §1.15].

$(\Rightarrow)$ Suppose that $Z(f^*) \neq cl_{\beta X} Z(f)$. Then there is a $z$-ultrafilter $U_p \in Z(f^*)$ but $U_p \notin cl_{\beta X} Z(f)$. This means that $Z(f) \notin U_p$. But then there is a zero-set $E$ in $U_p$ which does not meet $Z(f)$, and hence by [7, §1.15], $E$ is completely separated from $Z(f)$. But since $E \in U_p$ and $\lim_{U_p} f = 0$, it follows that $E$ is a near-zero-set of $f$ with respect to $C^*(X)$.

$(\Leftarrow)$ Suppose that $E$ is a near-zero-set of $f$ with respect to $C^*(X)$ and suppose that $E$ is completely separated from $Z(f)$. Then by [7, §1.15], there is a zero-set $F$ containing $E$ such that $F$ and $Z(f)$ are disjoint. As the property of being a near-zero-set of a function with respect to $A(X)$ is closed under supersets, $F$ is a near-zero-set of $f$ with respect to $C^*(X)$. Since $F$ is a near-zeros-set of $f$ with respect to $C^*(X)$, the function $f$ achieves values on $F$ arbitrarily close to zero. Thus the sets $\{ x : |f(x)| \leq 1/n \} \cap F$, for $n \in \mathbb{N}$, are nonempty zero-sets with the finite intersection property. So there is a $z$-ultrafilter $U_p$ containing these sets. Clearly $\lim_{U_p} f = 0$, so $p \in Z(f^*)$. On the other hand, since $Z(f)$ and $F$ are completely separated, $Z(f) \notin U_p$, and so $p \notin cl_{\beta X} Z(f)$. \qed

4. Closures of Zero-Sets of $X$ in $v_A X$

We now consider the problem of when the closure of a zero-set in $X$ is a zero-set in $v_A X$. It was shown by Rudd in [11, Example 1.5] that if $E \in Z[X]$, then it is possible that $cl_{\beta X} E \notin Z[\beta X]$. Hence it is not
always the case that the closure of a zero-set in $X$ is a zero-set in $v_A(X)$.
The following example is an alternative to Rudd’s example that fits well with the formalism of this paper.

**Example 4.1.** Let $A(X) = C^*([0, \infty))$ and let $f \in A(X)$ be any function whose zero-set is $E = \{1, 2, \ldots\}$. So $cl_{\beta X} Z(f)$ consists of those $\mathfrak{z}$-ultrafilters containing $E$. Choose a sequence $F = \{a_n : n = 1, 2, \ldots\}$ of positive numbers disjoint from $E$ such that $|a_n - n| < 1$ for each $n$ and $f(a_n) \to 0$ as $n \to \infty$. Since the $a_n$ are close to $n$, the set $F$ is unbounded, and hence $F$ is contained in some free $\mathfrak{z}$-ultrafilter $\mathcal{U}_p$. As $\mathcal{U}_p$ is free, $\lim_{\mathcal{U}_p} f = 0$. Hence by (2), $p \in Z(f^\beta)$. But $p \notin cl_{\beta X} Z(f)$ because $E \cap F = \emptyset$. So $cl_{\beta X} Z(f) \subsetneq Z(f^\beta)$ (strict containment). Since this is true for each $f$ with zero-set $E$, it follows that $cl_{\beta X} E$ is not a zero-set in $\beta X$. (Note that if there were a function $g \in C(\beta X)$ whose zero-set were $cl_{\beta X} E$, then the restriction of $g$ to $X$ would have zero-set $E$.)

In [14, §5], Swardson asked the following two questions. When is $cl_{\beta X} E$ a zero-set in $\beta X$ if $E$ is a zero-set of $X$? When does $cl_{\beta X} Z(f) = Z(f^\beta)$ for $f \in C^*(X)$? Some progress to answering these had already been given. For example in [15, Lemma 5], Terada found a characterization for when $cl_{\beta X} Z(f)$ is a zero-set of $\beta X$. For other results on the closures of zero-sets in $\beta X$ see [5]. We generalize these questions to $A(X)$, and compare the following two statements.

1. For every zero-set $E$ in $X$, its closure $cl_{v_A X} E$ is a zero-set in $v_A X$.
2. For every $f \in A(X)$ we have $cl_{v_A X} Z(f) = Z(f^{v_A})$.

Let $E$ be a zero-set in $X$. Statement (2) implies that for every $f \in A(X)$ with $Z(f) = E$ we have $cl_{v_A X} E = Z(f^{v_A})$ whereas Statement (1) follows if this equality holds only for some $f \in A(X)$ (so that $cl_{v_A X} E$ is a zero-set in $v_A X$). So Statement (2) clearly implies Statement (1). We have shown (Theorem 3.7) that the second statement characterizes $C(X)$ among intermediate $C$-rings. We now show that the first statement characterizes those $C$-rings $A(X)$ for which each $\mathfrak{z}$-filter is a $3_A$-filter. By a $3_A$-filter we mean a $\mathfrak{z}$-filter $\mathcal{F}$ with the property that

$$3_A 3_A^- [\mathcal{F}] = \mathcal{F}.$$  

Note that if $A(X) = C(X)$ then every $\mathfrak{z}$-filter is a $3_A$-filter because in this case $3_A(f) = \langle Z(f) \rangle$ for each $f$, and it is known that $ZZ^\perp[\mathcal{F}] = \mathcal{F}$ for every $\mathfrak{z}$-filter $\mathcal{F}$ ([7] p. 26)).

**Theorem 4.2.** Let $A(X)$ be an intermediate $C$-ring. The following are equivalent.

(a) Every $\mathfrak{z}$-filter on $X$ is a $3_A$-filter.
(b) For every zero-set $E$ in $X$, there exists $f \in A(X)$ such that $E = Z(f)$ and $\mathcal{Z}_A(f) = \langle Z(f) \rangle$.
(c) For every zero-set $E$ in $X$, there exists $f \in A(X)$ such that $E = Z(f)$ and no near-zero-set of $f$ with respect to $A(X)$ is completely separated from $Z(f)$.
(d) For every zero-set $E$ in $X$, its closure $\text{cl}_{v_A}E$ is a zero-set in $\text{v}_A X$.

Proof. $(a) \Rightarrow (b)$. Let $E$ be a zero-set in $X$, let $\mathcal{F} = \langle E \rangle$, and let $I = \mathcal{Z}_A[\mathcal{F}]$. By hypothesis, $\mathcal{Z}_A[I] = \mathcal{F}$. So there exists $f \in I$ such that $E \subseteq \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[\mathcal{Z}_A[f]] = \mathcal{F} = \langle E \rangle$; hence $\mathcal{Z}_A(f) = \langle E \rangle$.

$(b) \Rightarrow (c)$. By Lemma 3.6, we have $Z(f) = E$, and so $\mathcal{Z}_A(f) = \langle Z(f) \rangle$.

$(c) \Rightarrow (d)$. Let $E$ be a zero-set in $X$ and let $f \in A(X)$ such that $\mathcal{Z}(f) = E$ and $\mathcal{Z}_A(f) = \langle Z(f) \rangle$. Then, by Lemma 3.6, $\text{cl}_{v_A}Z(f) = Z(f^{v_A})$, and so $\text{cl}_{v_A}Z(f) = \text{cl}_{v_A}E$ is a zero-set in $\text{v}_A X$.

$(d) \Rightarrow (a)$. Let $E$ be a zero-set in $X$ with the property that $\text{cl}_{v_A}E$ is a zero-set in $\text{v}_A X$. Let $\tilde{f} \in C(\text{v}_A X)$, such that $\mathcal{Z}(\tilde{f}) = \text{cl}_{v_A}E$. Since $A(X)$ is a $C$-ring, by Theorem 2.4, $A(X)$ is isomorphic to $C(\text{v}_A(X))$ by the isomorphism $f \mapsto f^{v_A}$. Thus there exists $f$, such that $f^{v_A} = \tilde{f}$. It follows that $\mathcal{Z}(f) = E$ and $\mathcal{Z}(f^{v_A}) = \text{cl}_{v_A}Z(f)$. It follows by Lemma 3.6 that for this $f$ we have $\mathcal{Z}_A(f) = \langle Z(f) \rangle$. \hfill \□

5. Maximal Ideals under $\mathcal{Z}_A$

Recall that $\mathcal{Z}_A$ not only maps ideals in $A(X)$ to $z$-filters on $X$, but that $\mathcal{Z}_A$ extends the map $Z$ defined for $C(X)$ to all intermediate rings $A(X)$. This means that if $A(X) = C(X)$ then $\mathcal{Z}_A[I] = Z[I]$ for every ideal $I$ in $A(X)$. It is known that $Z$ maps each maximal ideal of $C(X)$ to a $z$-ultrafilter on $X$. We investigate the corresponding property for $\mathcal{Z}_A$ on any intermediate ring $A(X)$. It was shown in [9, Example 4.5] that if $M$ is a maximal ideal in $A(X)$ it does not necessarily follow that $\mathcal{Z}_A[M]$ is a $z$-ultrafilter on $X$. This raises the following question: does the property of $\mathcal{Z}_A$ mapping maximal ideals to $z$-ultrafilters characterize $C(X)$ among intermediate rings $A(X)$? We show that the answer to this question is negative by exhibiting intermediate rings $A(X)$, different from $C(X)$, for which the property does hold (Theorem 5.2). We begin by giving a sufficient condition for an intermediate ring $A(X)$ to have this property.
Theorem 5.1. Let $A(X)$ be an intermediate C-ring for which every $z$-filter on $X$ is a $3_A$-filter. If $M$ is a maximal ideal in $A(X)$, then $3_A[M]$ is a $z$-ultrafilter on $X$.

Proof. By Lemma 2.3(b), there is a unique $z$-ultrafilter $U$ such that $3_A[M] \subseteq U$. Now, let $E \in U$. Then by Theorem 4.2 (the fact that (a) implies (b)), there exists $f \in A(X)$ such that $3_A(f) = \langle E \rangle \subseteq U$. By definition of $3_A^{-}$, it holds that $f \in M$. Hence $E \in 3_A[M]$. □

A $P$-space is a completely regular Hausdorff space in which every zero-set is open. We show that if $A(X)$ is an intermediate ring on a $P$-space $X$, then $3_A$ does map maximal ideals to $z$-ultrafilters. In particular, if $X$ is a $P$-space, then $C^*(X)$ has this property, and so the property does not characterize $C(X)$ among intermediate rings.

Theorem 5.2. If $A(X)$ is an intermediate C-ring on a $P$-space $X$ and if $M$ is a maximal ideal in $A(X)$, then $3_A[M]$ is a $z$-ultrafilter on $X$.

Proof. Because $X$ is a $P$-space, for every zero-set $E$ in $X$, it holds that $E$ is open, and hence the characteristic function $f$ of the complement of $E$ is continuous (and hence in $A(X)$). By definition of $f$, we have that $Z(f) = E$, and it is easy to see from the definition of $3_A$ that $3_A(f) = \langle E \rangle$. Then by Theorem 4.2 (the fact that (b) implies (a)), every $z$-filter on $X$ is a $3_A$-filter. Then by Theorem 5.1 if $M$ is a maximal ideal in $A(X)$, then $3_A[M]$ is a $z$-ultrafilter on $X$. □

The following problem remains open.

Problem 5.3. Characterize those intermediate rings $A(X)$ for which $3_A[M]$ is a $z$-ultrafilters on $X$ whenever $M$ is a maximal ideal in $A(X)$.

Any characterization of the intermediate rings satisfying the condition in the problem must involve properties of both the topology of $X$ and the ring structure of $A(X)$. This is because from Theorem 5.2 we know that if $X$ is a $P$-space, then the condition holds for any intermediate ring $A(X)$, whereas it is well known that if $A(X) = C(X)$ then the condition holds for any completely regular topological space $X$.

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References

C(X) AMONG INTERMEDIATE C-RINGS ON X


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