# On Temporal Dynamic Epistemic Logic

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# Example: Loss and Recovery of Knowledge

Agents Ann, Bob, Cathy, and David have common knowledge that there are four cards: two indistinguishable ♠, one ♢, and one ♣. The cards are distributed so that Ann and Bob get ♠, Cathy ♢, and David ♣. There is the following conversation:

i. *Ann*: "I do not have ♢."

ii. *David*: "I do not have ♠."

iii. *Cathy*: "Before Ann's announcement, I of course knew that Bob did not know Ann's suit; but as a result of Ann's announcement, I no longer knew of Bob's uncertainty, and as a result of David's announcement, I know it again."

# What Temporal Dynamic Epistemic Logic (TDEL) has to offer

1. Ability to express knowledge of agents about objective facts and about knowledge of others (from Epistemic Logic).
2. Ability to express epistemic consequences of epistemic actions (from Dynamic Epistemic Logic).
3. Ability to express past and future knowledge.

When formally expressing Cathy's statement, we shall make use of all three of these.

# Some History

- J. Hintikka (1962), *Knowledge and Belief: an Introduction to the Logic of Two Notions*.
  (argued that logic for knowledge and belief could be based on modal logic)
- R.C. Moore (1985), A Formal Theory of Knowledge and Action, in *Formal Theories of the Commonsense World*.
  (combined knowledge and actions using modal logic)
- J.Y. Halpern (1987), Using Reasoning about Knowledge to Analyze Distributed Systems, in *Annual Review of Comp. Science*.
  (distributed systems involve message passing resulting in change of knowledge)
- J. Plaza (1989), *Logics of Public Communication*.
  (developed a logic expressing results of public announcements)
- A. Baltag & L. Moss (2004), Logics for Epistemic Programs, in *Synthese*.
  (combined epistemic logic with dynamic logic)

# Basic TDEL Language $\mathcal{L}_Y$

Given sets $\Phi$ of *atomic propositions*, $\mathcal{A}$ of *agents*, and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ of *action types*,

- The *sentences* (also called *formulas*) are

$$\top \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_A \varphi \mid \Box^*_{\mathcal{B}} \varphi \mid [\pi]\varphi \mid Y_{\Box}\varphi$$

- the *programs* are

$$\textit{skip} \mid \textit{crash} \mid \sigma_k, \psi_1, \ldots, \psi_n \mid \pi \sqcup \rho \mid \pi ; \rho$$

# Some statements in the language

Suppose there is one action type *Pub* (for truthful “public annoncement”):

$[Pub, p]\Box^*_{\mathcal{A}} p$
Reading: a truthful public announcement of $p$ will result in common knowledge that $p$ is true.

$Y_{\Diamond}\Box_A p$
Reading: $A$ knew $p$ previously

$[Pub, \Box_B p]\Box_A q$
Reading: a truthful announcement that $B$ knows $p$ will result in $A$ knowing $q$.

# Expressing the Conversation

Let $\mathcal{A} = \{A, B, C, D\}$, let $\Phi$ consist of pairs such as $(A\spadesuit)$ expressing that $A$ has a $\spadesuit$ card, and let $\Sigma = \{Pub\}$ ($Pub$ for "public announcement")
Define $BkAs \equiv \Box_B(A\spadesuit) \vee \Box_B(A\clubsuit) \vee \Box_B(A\diamondsuit)$, read "Bob knows Ann's suit". The conversation goes as follows:

i. $\neg(A\diamondsuit)$

ii. $\neg(D\spadesuit)$

iii. $Y_\diamond Y_\diamond(\Box_C\neg BkAs \wedge [Pub, \neg(A\diamondsuit)](\neg\Box_C\neg BkAs \wedge [Pub, \neg(D\spadesuit)]\Box_C\neg BkAs))$

# State Models (also known as Epistemic Models)

**Definition (State Model)**

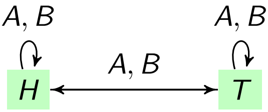
Fix a set $\Phi$ of "atomic propositions" and a set $\mathcal{A}$ of agents. A *state model* is a tuple $\mathbf{S} = (S, \xrightarrow{\mathcal{A}}, \|\cdot\|)$, where

1. $S$ is a set
2. $\xrightarrow{A}$ is an "epistemic" binary relation over $S$, that is $s \xrightarrow{A} t$ means $A$ considers $t$ possible at state $s$.
3. $\|\cdot\|$ is function from $\Phi$ to $\mathcal{P}(S)$.

Often $\xrightarrow{A}$ is an equivalence relation, and is thus called an "indistinguishability relation".

# Concealed coin example

Suppose that a coin is flipped. Two agents $A$ and $B$ do not know the result of the flip. $\Phi = \{h, t\}$ and both $\|h\| = \{H\}$ and $\|t\| = \{T\}$.



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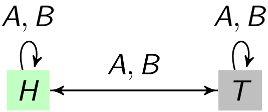


If it is revealed to both agents that result of the flip was heads, then T is removed

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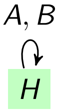


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# Muddy Children Example

Three brilliant children go to the park to play. When their father comes to find them, he sees that all three of them have mud on their foreheads. He then says, "At least one of you has mud on your forehead", and then asks, "Do you know if you have mud on your forehead?" The children simultaneously respond, "No".

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The father repeats his question, "Do you know if you have mud on your forehead?" The children again simultaneously respond "No".

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The father asks a third time "Do you know if you have mud on your forehead?", and this time all three children simultaneously answer, "Yes, I have!".

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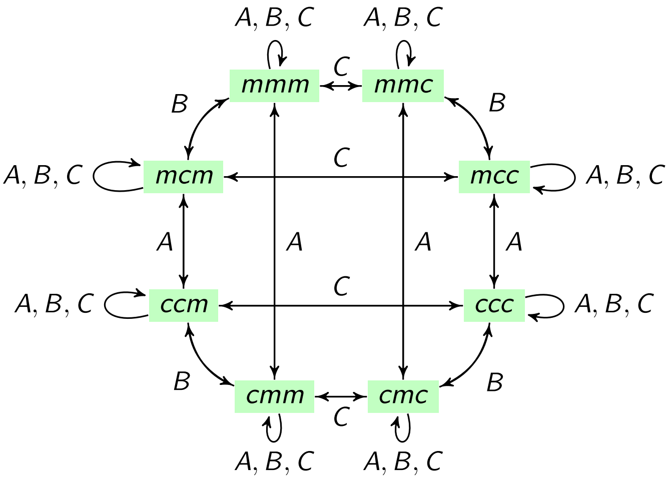

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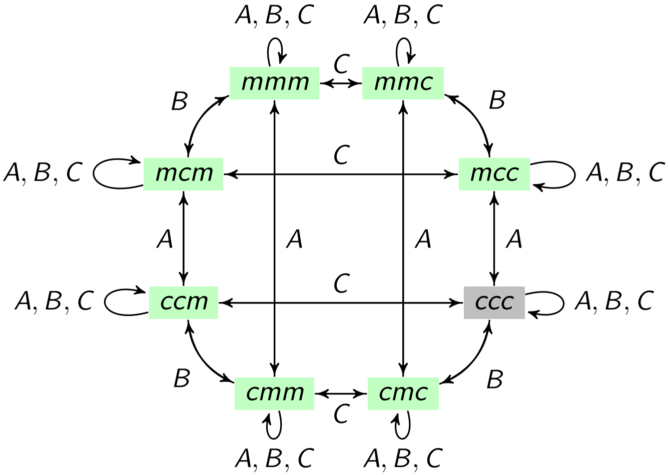


The father announces “at least one of you is muddy”

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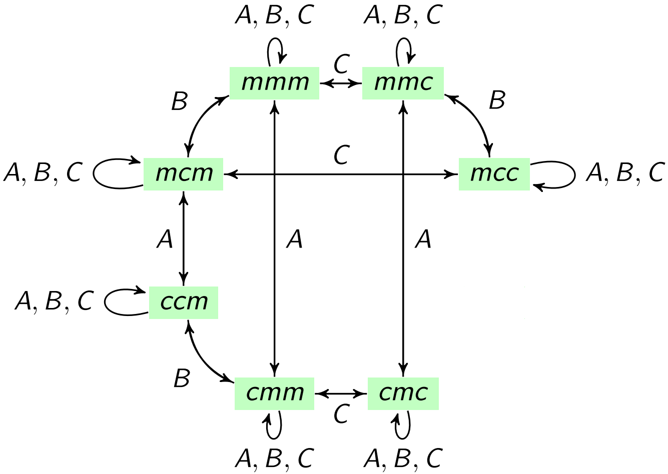


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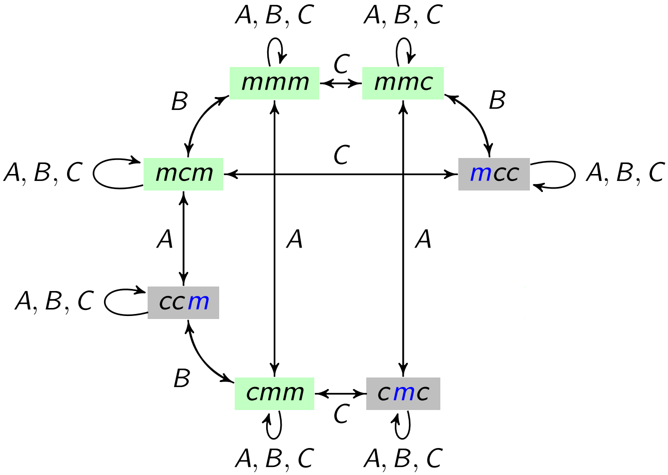


The children answer “No” for first time

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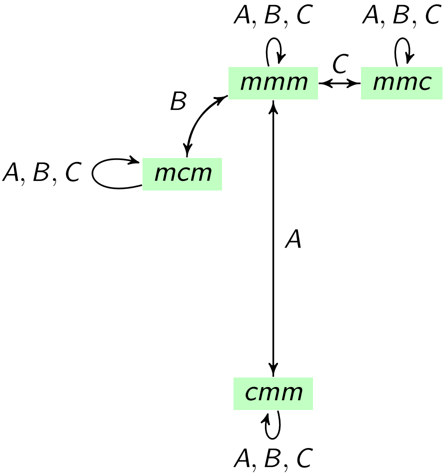


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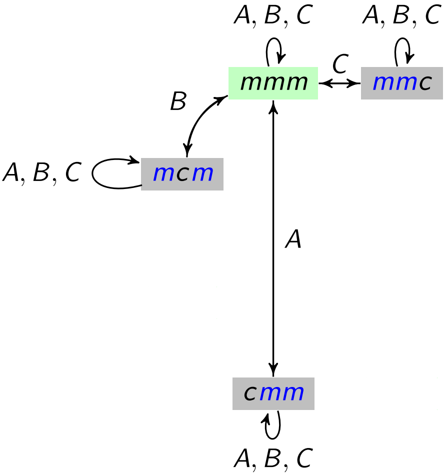


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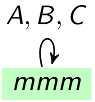


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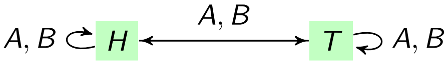
# Concealed coin and Private Announcement



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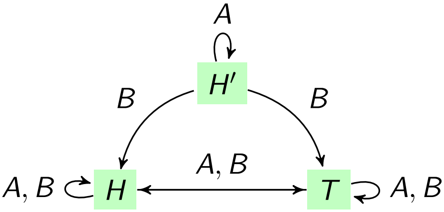



It is revealed to agent $A$ that result of the flip was heads, but $B$ does not know anything happened.

# Actions

The process of obtaining a new state model from an old one is formalized by an "update product" between a state model and an action. An action is characterized by an action model:

**Definition (Action Model)**

Fix a set $\mathcal{A}$ of agents. An *action model* is a triple $(\Sigma, \xrightarrow{\mathcal{A}}, \mathrm{pre})$

1. $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is an finite set of "actions" with a fixed enumeration.
2. $\xrightarrow{A}$ is a binary relation over $\Sigma$, where $\sigma \xrightarrow{A} \tau$ means $A$ considers $\tau$ a possible occurrence if $\sigma$ actually occurred.
3. $\mathrm{pre}$ is a function mapping each $\sigma \in \Sigma$ to class of model/state pairs $(\mathcal{S}, s)$, where $s$ is a state in model $\mathcal{S}$.

The function $\mathrm{pre}$ gives a precondition to each $\sigma$ in that $(\mathcal{S}, s) \in \mathrm{pre}(\sigma)$ if and only if $\sigma$ could occur in $(\mathcal{S}, s)$.

# Update Product

**Definition (Update Product)**

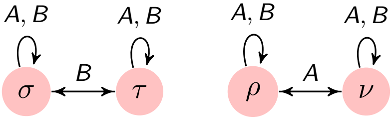
Given a state model $\mathbf{S} = (S, \xrightarrow{\mathcal{A}}, \|\cdot\|)$, and an action model $(\Sigma, \xrightarrow{\mathcal{A}}, \mathrm{pre})$, the *update product* is the state model $(S_\otimes, \xrightarrow{\mathcal{A}}_\otimes, \|\cdot\|_\otimes)$:

1. $S_\otimes = \{(s, \sigma) \in S \times \Sigma : (\mathbf{S}, s) \in \mathrm{pre}(\sigma)\}$
2. $(s, \sigma) \xrightarrow{A}_\otimes (t, \tau)$ iff $(s \xrightarrow{A} t$ and $\tau \xrightarrow{A} \sigma)$
3. $\|p\|_\otimes = \{(s, \sigma) \in S_\otimes : s \in \|p\|\}$

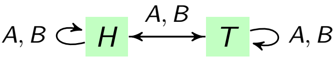
The first two components are fixed, and are called an Action Signature. $\mathrm{pre}$ may be specified by the list $(\mathrm{pre}(\sigma_1), \ldots, \mathrm{pre}(\sigma_n))$, and the update product is between a state model and this list.

# Semi-Private Announcement

An action signature:




After a coin is flipped



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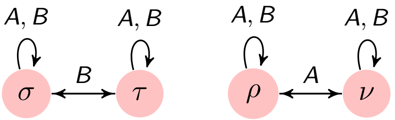


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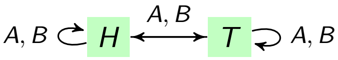
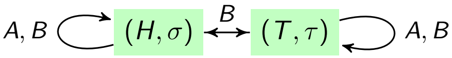



Semi-private announcement to $A$: preconditions $([\![h]\!], [\![t]\!], \emptyset, \emptyset)$.



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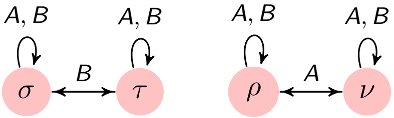


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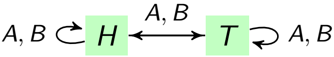


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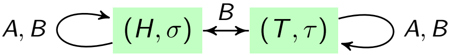



Semi-private announcement to $B$: preconditions $(\emptyset, \emptyset, [\![h]\!], [\![t]\!])$.

# Axioms

Language: $\mathcal{L}_A = \mathcal{L}_Y + \sigma$. Add to proof system of DEL the following axiom schema:

- $Y_\Box(\varphi \to \psi) \to (Y_\Box\varphi \to Y_\Box\psi)$ (normality)
- $Y_\Box p \leftrightarrow (Y_\Diamond\top \to p)$ (atomic permanence)
- $Y_\Diamond\varphi \leftrightarrow (\neg Y_\Box\bot \wedge Y_\Box\varphi)$ (partial functionality)
- $[\sigma_i\vec{\psi}]Y_\Box\varphi \leftrightarrow (\psi_i \to \varphi)$ (action-yesterday)
- $(Y_\Diamond\top \to \Box_A Y_\Diamond\top) \wedge (Y_\Box\bot \to \Box_A Y_\Box\bot)$ (initial-time)
- $Y_\Box\Box_A\varphi \to \Box_A Y_\Box\varphi$ (epistemic-yesterday)
- $\sigma \to \Box_A\neg\tau$ (where $\neg(\sigma \xrightarrow{A} \tau)$) (restriction)
- $Y_\Diamond\top \leftrightarrow \bigvee\{\sigma : \sigma \in \Sigma\}$
- $\sigma \to \neg\tau$ for each $\sigma \neq \tau$.
- $[\sigma_i\vec{\psi}]\sigma_i$

# Rules

Add to proof system of DEL the following rule:

- Necessitation:

$$\frac{\vdash \varphi}{\vdash Y_{\Box}\varphi}$$

# DEL Axioms

- $[\pi](\varphi \to \psi) \to ([\pi]\varphi \to [\pi]\psi)$ (normality)
- $\Box_A(\varphi \to \psi) \to (\Box_A\varphi \to \Box_A\psi)$ (normality)
- $\Box^*_{\mathcal{B}}(\varphi \to \psi) \to (\Box^*_{\mathcal{B}}\varphi \to \Box^*_{\mathcal{B}}\psi)$ (normality)
- $[\sigma_i\vec{\psi}]p \leftrightarrow (\psi_i \to p)$ (atomic permanence)
- $[\sigma_i\vec{\psi}]\neg\chi \leftrightarrow (\psi_i \to \neg[\sigma_i\vec{\psi}]\chi)$ (partial functionality)
- $[\sigma_i\vec{\psi}]\Box_A\varphi \leftrightarrow (\psi_i \to \bigwedge\{\Box_A[\sigma_j\vec{\psi}]\varphi : \sigma_i \xrightarrow{A} \sigma_j\})$ (Epis. action).
- $\Box^*_{\mathcal{B}}\varphi \to \varphi \wedge \bigwedge\{\Box_A\Box^*_{\mathcal{B}}\varphi : A \in \mathcal{B}\}$ (Epistemic-mix).
- $[skip]\varphi \leftrightarrow \varphi$
- $[crash]\bot$
- $[\pi;\rho]\varphi \leftrightarrow [\pi][\rho]\varphi$
- $[\pi \sqcup \rho]\varphi \leftrightarrow [\pi]\varphi \wedge [\rho]\varphi$

# DEL Rules

- Modus Ponens

$$\frac{\vdash \varphi \quad \vdash \varphi \to \psi}{\vdash \psi}$$

- Necessitation

$$\frac{\vdash \varphi}{\vdash \Box_A \varphi} \qquad \frac{\vdash \varphi}{\vdash \Box^*_{\mathcal{B}} \varphi} \qquad \frac{\vdash \varphi}{\vdash [\pi]\varphi}$$

- Action Rule: Let $\psi$ be a sentence, let $\alpha \in \Omega$, and let $\mathcal{B}$ be a set of agents. Let there be sentences $\chi_\beta$ for all $\beta$ such that $\alpha \xrightarrow{\mathcal{B}^*}_\Omega \beta$ (including $\alpha$ itself), and such that
  1. $\vdash \chi_\beta \to [\beta]\psi$
  2. if $A \in \mathcal{B}$ and $\beta \xrightarrow{A}_\Omega \gamma$, then $\vdash (\chi_\beta \wedge \mathrm{PRE}(\beta)) \to \Box_A \chi_\gamma$.

  From these assumptions, infer $\vdash \chi_\alpha \to [\alpha]\Box^*_{\mathcal{B}}\psi$.

# Provable formulas

- Extended partial functionality

$$\vdash [\alpha]\neg\varphi \leftrightarrow (\textsc{Pre}(\alpha) \rightarrow \neg[\alpha]\varphi)$$

- Extended atomic Permanence

$$\vdash [\alpha]p \leftrightarrow (\textsc{Pre}(\alpha) \rightarrow p)$$

  Proved using extended partial functionality and atomic permanence axiom

- Negative atomic permanence

$$\vdash Y_{\Box}\neg p \leftrightarrow (Y_{\Diamond}\top \rightarrow \neg p)$$

  Proved using atomic permanence and partial functionality

# Provable formulas

- Extended Action-yesterday

$$\vdash [\alpha][\sigma_i \vec{\psi}] Y_{\Box}\varphi \leftrightarrow (\langle\alpha\rangle\psi_i \rightarrow [\alpha]\varphi)$$

  Proved using extended partial functionality and action-yesterday axiom

- Extended Epistemic-mix

$$\vdash [\alpha]\Box_A\varphi \leftrightarrow (\mathrm{P{\scriptsize RE}}(\alpha) \rightarrow \bigwedge\{\Box_A[\beta]\varphi : \alpha \xrightarrow{A}_{\Omega} \beta\})$$

# Provable formulas

- Extended initial-non-initial time

$$\vdash Y^n_{\Box}\bot \rightarrow \Box_A Y^n_{\Box}\bot$$
$$\vdash Y^n_{\Diamond}\top \rightarrow \Box_A Y^n_{\Diamond}\top$$

  Proved using epistemic initial and non-initial time axioms as well as epistemic-yesterday.

- Action types

$$\vdash [\sigma_i\vec{\psi}]\sigma_j \leftrightarrow \neg\psi_i.$$

  Proved using $[\sigma_i\vec{\psi}]\sigma_i$ and $\sigma_i \rightarrow \neg\sigma_j$

# (Weak) Completeness

**Theorem (Sack)**

*If $\varphi \in \mathcal{L}_A$ is valid, then $\vdash \varphi$.*

Proof involves:

- Term rewriting to show that every formula is provably equivalent to a formula in some normal form (a form with limited occurrences of actions)
- Filtration in non-standard semantics
- Modifications of the filtration to a model isomorphic to a history (object of standard semantics). Modifications include
  - partial unravelling of the filtration into a tree-like structure
  - trimming of the model
  - addition of relational connections

# Histories vs Non-standard structures

- The structures described by $\mathcal{L}_A$ are histories: a sequence of state models, each state model an update product the previous with some action model.
- Non-standard semantics can be defined for normal form formulas.

## Definition

A non-standard structure $\mathcal{S} = (S, \xrightarrow{\mathcal{A}}, Y, g, \|\cdot\|)$

- $Y$ is a relation where $sYt$ indicates that $t$ was the case immediately before $s$.
- $g$ is a function assigning each state to an action in $\Sigma$ or $\emptyset$

Non-standard structures are easier to establish a filtration for.

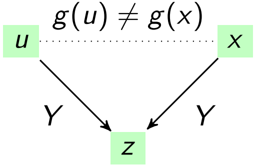
# Non-standard History

A structure $\mathcal{S} = (S, \xrightarrow{\mathcal{A}}, Y, g, \| \cdot \|)$ is called a *non-standard history* if it has the following properties:

1. Action types condition: $g(x) \neq \emptyset$ iff there exists $z$ such that $xYz$
2. Partial functionality of $Y$: If $xYz$ and $xYz'$ then $z = z'$
3. Bounded age: There exists $N$ such that for all $x$ there is no $z$ for which $xY^N z$.
4. Synchronicity: if $x \xrightarrow{A} z$, then for each $n$, $xY^n x'$ for some $x'$ iff $zY^n z'$ for some $z'$.

# Non-standard History cont'd

5. Update product states condition $a$: If $uYz$, $xYz$, and $x \neq u$, then $g(u) \neq g(x)$.
   To picture this, if $u \neq x$ then

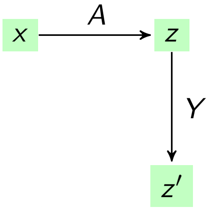



6. Update product states condition $b$: For each action type $\sigma \in \Sigma$ and each $n > 0$, there exists a formula $\varphi \in \mathcal{NF}$ such that for every $x$ for which $xY^nz$ for some $z$ and $xY^{n+1}z$ for no $z$,

$$\mathcal{S}, x \vDash \varphi \text{ iff there is a } u \text{ such that } g(u) = \sigma \text{ and } uYx.$$

# Non-standard History cont'd

7. Update product relation condition $a$: If $x \xrightarrow{A} zYz'$, there exists an $x'$ such that $xYx' \xrightarrow{A} z'$.




8. Update product relation condition $b$: If $x \xrightarrow{A} z$, and $g(x) \neq \emptyset$, then $g(x) \xrightarrow{A} g(z)$.

# Non-standard History cont'd

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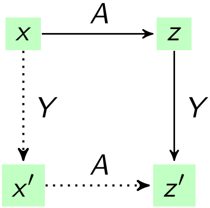
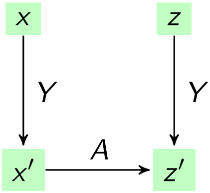



8. Update product relation condition $b$: If $x \xrightarrow{A} z$, and $g(x) \neq \emptyset$, then $g(x) \xrightarrow{A} g(z)$.

# Non-standard History cont'd

9. Update product relation condition $c$: If $xYx'$, $zYz'$, $x' \xrightarrow{A} z'$, and $g(x) \xrightarrow{A} g(z)$, then $x \xrightarrow{A} z$.
   To picture this, if $g(x) \xrightarrow{A} g(z)$




10. Update product valuation condition: If $xYz$, then $\mathcal{S}, x \vDash p$ iff $\mathcal{S}, z \vDash p$.

# Non-standard History cont'd

9. Update product relation condition $c$: If $xYx'$, $zYz'$, $x' \xrightarrow{A} z'$, and $g(x) \xrightarrow{A} g(z)$, then $x \xrightarrow{A} z$.
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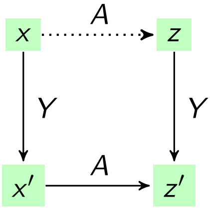



10. Update product valuation condition: If $xYz$, then $\mathcal{S}, x \vDash p$ iff $\mathcal{S}, z \vDash p$.

# Filtration and Modification

The numbers 1 through 10 correspond to the 10 properties of a non-standard history

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| filtration | Y | N | N | N | N | N | N | Y | N | Y |
| unravelling | Y | N | N | N | Y | N | N | Y | Y | Y |
| trimming | N | Y | Y | Y | Y | N | N | Y | Y | Y |
| modifying $g$ | Y | Y | Y | Y | Y | N | N | Y | Y | Y |
| expanding $\xrightarrow{A}$ | Y | Y | Y | Y | Y | N | Y | Y | Y | Y |
| modifying $\|\cdot\|$ | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y |

Y means that the model has the property

N means that the property is not guaranteed

# Normal Form

Normal Form $\mathcal{NF}$ is a sublanguage of $\mathcal{L}_A$ restricting occurrences of programs to $[\alpha]\Box^*_{\mathcal{B}}\varphi$ and only permitting programs of the form

$$\alpha = (\cdots((\sigma_1\vec{\psi}^1; \sigma_2\vec{\psi}^2)\cdots)\sigma_k\vec{\psi}^k).$$

# Layered sets

After the trimming stage, we will make use of layered sets for layered truth lemmas.

Fix a consistent formula $\varphi$, and let $n$ be its yesterday depth $\delta\epsilon(\varphi)$. For each $k \leq n$, $f_k : \mathcal{NF} \to \mathcal{P}(\mathcal{NF})$ is a function with many desirable properties. For $1 \leq k \leq n$,

$$X_k = \bigcup \{ f_j(\chi) : \chi \in s(\varphi), \delta\epsilon(\chi) = j \},$$

where $s(\varphi)$ is the set of subformulas of $\varphi$. It turns out that $X_n = f_n(\varphi)$.

# Filtration

Given a maximal consistent set of formulas $U$, let $U^* = \bigwedge\{\psi : \psi \in U \cap X_n\} \wedge \bigwedge\{\neg\psi : \psi \in X_n - U\}$. (It is helpful to assume a fixed order on $X_n$.) Maximal consistent sets $U$ and $V$ are equivalent ($U \equiv V$) iff $U^* = V^*$.

## Definition

Define filtration $\mathcal{S}_F = (S, \xrightarrow{\mathcal{A}}, Y, g, \|\cdot\|)$:

- $S$ is the set of $\equiv$-equivalence classes
- $[U] \xrightarrow{A} [V]$ iff whenever $\Box_A\psi \in U \cap X_n$, $\psi \in V$
  $[U]Y[V]$ iff whenever $Y_\Box\psi \in U \cap X_n$, $\psi \in V$
- $g : S \to \Sigma \cup \{\emptyset\}$ is defined by $g([U]) = \sigma$ iff $\sigma \in U \cap X_n$.
  $\|p\| = \{[U] : p \in U \cap X_n\}$.

# Strong Truth Lemma

**Lemma (strong truth lemma)**

*(extends DEL truth lemma) For each $\chi \in X_n$ and equivalence class $[U] \in S$, $\chi \in U$ iff $\mathcal{S}_F, [U] \models \chi$.*

Recall that this filtration is not guaranteed to be a non-standard history. Among the properties not established:

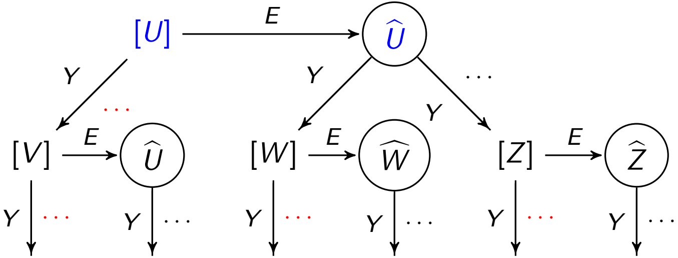
5. If $xYz$, $x'Yz$, and $x \neq x'$, then $g(x) \neq g(x')$.
9. If $xYx' \xrightarrow{A} z'$ and there exists $z$ such that $zYz'$ and $g(x) \xrightarrow{A} g(z)$, then $x \xrightarrow{A} z$.

# partial unraveling

Let $S_F$ be the set in $\mathcal{S}_F$, and suppose $\varphi \in U$. Let $S_U$ be the smallest set such that:

1. $[U] \in S_U$.
2. $zE[W] \in S_U$ if $z \in S_U$ and either
   (a) $z = [U]$ and $[U] \xrightarrow{A}_F [W]$ for some $A \in \mathcal{A}$.
   (b) $zE[V] \in S_U$ with $[V] \in S_F$ where $[V] \xrightarrow{A} [W]$ for some $A \in \mathcal{A}$.
   (c) $z = xY[V]$, with $x \in S_U$ and $[V] \xrightarrow{A} [W]$ for some $A \in \mathcal{A}$
3. $zY[W] \in S_U$ if $z \in S_U$ and either
   (a) $z = xE[V]$, with $x \in S_U$ and $[V]Y[W]$ $z = xY[V]$, with $x \in S_U$ and $[V]Y[W]$

Define a new model $\mathcal{S}_U$, where the relations $Y$ and $\xrightarrow{A}$ follow the structure.




where for each $V$, $\widehat{V}$ is (isomorphic to) the submodel of $\mathcal{S}_F$ consisting of states epistemically reachable from $[V]$. The $E$ above the arrows is suggestive of the names of the states, and each $\xrightarrow{E}$ corresponds to possibly many relational connections for each agent.

# Truth Lemma

For each $x \in S_U$, let $C(x)$ be the last equivalence class represented by $x$.

**Lemma**

*For each $\chi \in X_n$ and element $x \in S_U$, $\chi \in C(x)$ iff $\mathcal{S}_U, x \vDash \chi$.*

This mostly comes from modal bisimulation.

# Trimming

Given any state $x \in S_U$, let $\delta\iota(x)$ be the number of $Y$ relational steps from $U$. Obtain a new modal $\mathcal{S}_T$ by removing all states $x$ for which $\delta\iota(x) > n$. Set $g(x) = \emptyset$ for all $x$ for which $\delta\iota(x) = n$.

# Closure conditions

## Lemma (Closure Conditions)

*The following conditions hold:*

1. *For each* $x \in S_T$, *if* $\psi_1$ *and* $\psi_2$ *are such that* $\mathcal{S}_U, x \vDash \psi_i$ *iff* $\mathcal{S}_T, x \vDash \psi_i$ *for* $i = 1, 2$, *then* $\mathcal{S}_U, x \vDash \psi_1 \wedge \psi_2$ *iff* $\mathcal{S}_T, x \vDash \psi_1 \wedge \psi_2$.
2. *For each* $x \in S_T$, *if* $\psi$ *is such that* $\mathcal{S}_U, x \vDash \psi$ *iff* $\mathcal{S}_T, x \vDash \psi$, *then* $\mathcal{S}_U, x \vDash \neg\psi$ *iff* $\mathcal{S}_T, x \vDash \neg\psi$
3. *If for all* $x$ *for which* $\delta\iota(x) = j \leq n$, $\mathcal{S}_U, x \vDash \psi$ *iff* $\mathcal{S}_T, x \vDash \psi$, *then for each* $x$ *with yesterday distance* $j$, $\mathcal{S}_U, x \vDash \Box_A\psi$ *iff* $\mathcal{S}_T, x \vDash \Box_A\psi$.

**Lemma (Layered Truth Lemma)**

*For each* $\chi \in X_j$ *and element* $x \in S_T$ *with* $\delta\iota(x) = n - j$, $\chi \in C(x)$ *iff* $\mathcal{S}_T, x \vDash \chi$.

IH: for each $i < j$, if $\chi \in X_i$ and $\delta\iota(x) = n - i$, then $\mathcal{S}_T, x \vDash \chi$ iff $\mathcal{S}_U, x \vDash \chi$.

Base Case ($j = 0$): Internal induction on some appropriate order $<$ on formulas.

IH: for all $\chi < \psi$, if $\chi \in X_0$ and $\delta\iota(x) = n$, then $\mathcal{S}_T, x \vDash \chi$ iff $\mathcal{S}_U, x \vDash \chi$.

Base Case: $\top$ and $p \in \Phi$ ($\sigma \notin X_0$).

Inductive Step: $\neg$, $\wedge$, $\Box_A$, $\Box^*_{\mathcal{B}}$, $[\alpha]\Box^*_{\mathcal{B}}$ ($Y_\Box\chi \notin X_0$).

Inductive step $j > 0$: Internal induction on some appropriate order $<$ on formulas.

IH: for all $\chi < \psi$, if $\chi \in X_j$ and $\delta\iota(x) = n - j$, then $\mathcal{S}_T, x \vDash \chi$ iff $\mathcal{S}_U, x \vDash \chi$.

Base Case: $\top$ and $p \in \Phi$, $\sigma \in \Sigma$.

Inductive Step: $\neg$, $\wedge$, $\Box_A$, $\Box^*_{\mathcal{B}}$, $[\alpha]\Box^*_{\mathcal{B}}$, $Y_\Box$.

$\top$, $p$: mostly from definition of semantics: $\mathcal{S}, x \models p$ iff $x \in \|p\|$. $\sigma$:

mostly from definition of semantics: $\mathcal{S}, x \models \sigma$ iff $g(x) = \sigma$. $\neg$, $\wedge$,

$\Box_A$, $\Box^*_{\mathcal{B}}$: from closure conditions. $[\alpha]\Box^*_{\mathcal{B}}$: mostly from definition of

semantics: $\mathcal{S}, x \not\models [\alpha]\Box^*_{\mathcal{B}}\psi$ iff there exists a $\mathcal{B}^*$ path from $x$ with desired properties. $Y_\Box$: makes use of the the property: if

$Y_\Box\psi \in X_{i+1}$, then $\psi$ is a subformula of $\neg\Box_A\chi$, where $\chi$ is a boolean combination of formulas in $X_i$. Apply the IH to formulas in $X_i$ at all states with yesterday distance $n - i$, and then use closure conditions.

# Recent Related Work

- J. Sack, Temporal Languages for Epistemic Programs, to appear in *Logic, Language and Information*.
  (Introduces a variety of temporal additions to DEL)
- P. Balbiani, A. Baltag, H.P. van Ditmarsch, A. Herzig, T. Hoshi, T. de Lima (2007), What can we achieve by arbitrary announcements? — A dynamic take on Fitch's knowability, In *TARK Proceedings*.
  (Proved completeness for language that adds next-time operator to Public Announcement Logic)
- H.P. van Ditmarsch, W. van der Hoek & B.P. Kooi (2007), *Dynamic Epistemic Logic*.
  (Textbook on DEL)

# Further work

- Extend completeness of next-time operator to DEL case
- Past (from previous time to always in the past)
- Connect with Temporal Epistemic Logic
- Temporal Probabilistic Dynamic Epistemic Logic (extend Kooi's "Probabilistic Dynamic Epistemic Logic")
- Game theory (DEL has been used to describe beliefs in games: making a move in a game can be viewed as a public announcement, but so far, belief revision is lacking)