Modal Logic for Mixed Strategies in Games

Joshua Sack and Wiebe van der Hoek

August 14, 2014

Game

Definition (Game)

A (strategic form) game is a tuple

$$\mathcal{G} = (Ag, \{\Pi_i\}_{i\in Ag}, \{u_i\}_{i\in Ag}),$$

$\mathcal{G}$ where

- Ag - finite set of players,
- Π_i - finite set of pure strategies for agent $i \in Ag$, and
- $u_i : \Pi \to \mathbb{Q}$ - utility function, assigning agent i's payoff to each pure strategy profile,

Definition (Pure profile)

A pure strategy profile is a tuple

$$(\pi_1, \ldots, \pi_n),$$

such that each $\pi_i \in \Pi_i$. Let Π be all pure strategy profiles.

Mixed strategies

Definition (Mixed strategy)

Π_i is agent i's set of pure strategies. A mixed strategy is a probability mass function

$$\sigma_i : \Pi_i \to [0, 1],$$

Let Σ_i be all mixed strategies for player i.

Definition (Mixed profile)

A mixed (strategy) profile tuple is a tuple $(\sigma_1, \ldots, \sigma_n)$ for each player $i \in Ag$.

Definition (Mixed strategy function)

A mixed strategy function is a function $\sigma : \Pi \to [0, 1]$, where Π is the set of pure strategy profiles.

Correlated profiles

There exist mixed strategy functions that are not equivalent to mixed profiles: correlated strategies.

Example

σ	H_b	T_b
H_a	0.2	0.2
T_a	0.2	0.4

Here, whether a chooses H_a with probability $1/2$ or probability $1/3$ depends on b's strategy.

We will call a *mixed strategy function* a *mixed profile* only if the mixed strategy function is equivalent to a mixed profile (and are hence uncorrelated).

Expected utility

The utility function $u_i : \Pi \to \mathbb{R}$ can be extended from pure to mixed strategy profiles by

$$u_i(\sigma) = \sum_{\pi \in \Pi} \sigma(\pi) u_i(\pi).$$

Nash equilibrium

Given a mixed strategy profile σ and a mixed strategy ρ_i, denote by (ρ_i, σ_{-i}) the strategy profile

$$(\sigma_1, \ldots, \sigma_{i-1}, \rho_i, \sigma_{i+1}, \ldots, \sigma_n).$$

Definition

A Nash equilibrium is a (mixed or pure) strategy profile σ such that for any agent i and any strategy ρ_i

$$u_i(\sigma) \geq u_i(\rho_i, \sigma_{-i}).$$

Nash equilibria in matching pennies

- $Ag = \{a, b\}$
- $\Pi_i = \{H_i, T_i\}$ (heads and tails of player i's coin)
- $u_i : \Pi \to \{-1, 1\}$ is given by the following chart:

	H_b	T_b
H_a	$+1, -1$	$-1, +1$
T_a	$-1, +1$	$+1, -1$

- Pure strategy Nash equilibria: none
- Mixed strategy Nash equilibria: one
  each player plays $1/2$ for both their strategies.
  [Corresponding mixed strategy function: uniform probability function (each of the 4 pure profiles gets probability $\frac{1}{4}$)]

Background on some game logics

To give a sense of the diversity of game logics:

- Marc Pauly's dissertation, "Logic for social software", ILLC, University of Amsterdam, 2001.
  defines "Game logic" and "Coalition logic" with formulas describing what certain (groups of) agents can enforce.
- R. Alur, T. Henzinger, O. Kupferman. Alternating-Time temporal logic. Journal of the ACM. 2002.
  describes powers of coalitions over time, using concurrent game models.
- J. Halpern: Substantive Rationality and Backward Induction. *Games and Economic Behavior*, 37:425-435, 2001.
  gives a logic for a fixed game that is defined on Kripke models whose states are labelled with (pure) strategy profiles.

Language

Given a game $\mathcal{G}$, let

$$t ::= aP(\boldsymbol{\pi}) \mid t + t$$
$$\varphi ::= t \geq a \mid \neg\varphi \mid \varphi \wedge \varphi \mid [G]\varphi \mid [\sqsubset_i]\varphi \mid [\sqsupset_i]\varphi \mid [=_i]\varphi$$

where $a \in \mathbb{Q}$, $\pi \in \Pi$, $i \in Ag$, and $G \subseteq Ag$.

Fix $\widetilde{\Sigma} \subseteq \Sigma$ - a set of mixed profiles (as functions). Formulas evaluated (only) on $\sigma, \tau \in \widetilde{\Sigma}$ (relations restricted to $\widetilde{\Sigma} \times \widetilde{\Sigma}$).

$\sigma \vDash \sum_{k=1}^{n} q_k P(\boldsymbol{\pi}_k) \geq r$	iff	$\sum_{k=1}^{n} q_k \sigma(\pi_k) \geq r$
$\sigma \vDash [G]\varphi$	iff	$\tau \vDash \varphi$ whenever for each $i \in G$, $\sigma_i = \tau_i$
$\sigma \vDash [\sqsubset_i]\varphi$	iff	$\tau \vDash \varphi$ whenever $u_i(\sigma) < u_i(\tau)$
$\sigma \vDash [\sqsupset_i]\varphi$	iff	$\tau \vDash \varphi$ whenever $u_i(\sigma) > u_i(\tau)$
$\sigma \vDash [=_i]\varphi$	iff	$\tau \vDash \varphi$ whenever $u_i(\sigma) = u_i(\tau)$

- J. Sack & W. van der Hoek. Modal Logic for Mixed Strategies. *Studia Logica*, 2014.

Language

Given a game $\mathcal{G}$, let

$$t ::= aP(\boldsymbol{\pi}) \mid t + t$$
$$\varphi ::= t \geq a \mid \neg\varphi \mid \varphi \wedge \varphi \mid [G]\varphi \mid [\sqsubset_i]\varphi \mid [\sqsupset_i]\varphi \mid [=_i]\varphi$$

where $a \in \mathbb{Q}$, $\pi \in \Pi$, $i \in Ag$, and $G \subseteq Ag$.

Fix $\widetilde{\Sigma} \subseteq \Sigma$ - a set of mixed profiles (as functions). Formulas evaluated (only) on $\sigma, \tau \in \widetilde{\Sigma}$ (relations restricted to $\widetilde{\Sigma} \times \widetilde{\Sigma}$).

$\sigma \models \sum_{k=1}^{n} q_k P(\boldsymbol{\pi}_k) \geq r$	iff	$\sum_{k=1}^{n} q_k \sigma(\pi_k) \geq r$
$\sigma \models [G]\varphi$	iff	$\tau \models \varphi$ whenever for each $i \in G$, $\sigma_i = \tau_i$
$\sigma \models [\sqsubset_i]\varphi$	iff	$\tau \models \varphi$ whenever $u_i(\sigma) < u_i(\tau)$
$\sigma \models [\sqsupset_i]\varphi$	iff	$\tau \models \varphi$ whenever $u_i(\sigma) > u_i(\tau)$
$\sigma \models [=_i]\varphi$	iff	$\tau \models \varphi$ whenever $u_i(\sigma) = u_i(\tau)$

- J. Sack & W. van der Hoek. Modal Logic for Mixed Strategies. *Studia Logica*, 2014.

Meaning of $[G]$

- $[G]\varphi$ means that
  φ is true at any mixed profile where those not in G potentially switch to different strategies.
- $[Ag \setminus \{i\}]\varphi$ means that
  φ is true whenever i potentially switches to a different strategy.
- $[\emptyset]\varphi$ means that
  φ is true at all profiles.

Abbreviations

Expressing that τ is the mixed profile:

$$\boldsymbol{\tau} \stackrel{\text{def}}{=} \bigwedge_{\pi \in \Pi} P(\boldsymbol{\pi}) = \tau(\pi)$$

The utility for i is term

$$\mathbf{u}_i \stackrel{\text{def}}{=} \sum_{\pi \in \Pi} u_i(\pi) P(\boldsymbol{\pi})$$

The probability i has for playing pure strategy π_i is a term:

$$P_i(\boldsymbol{\pi}_i) \stackrel{\text{def}}{=} \sum \{P(\boldsymbol{\rho}) \mid \rho \in \Pi, \rho_i = \pi_i\}$$

Expressing that τ_i is i's (mixed) strategy

$$\boldsymbol{\tau}_i \stackrel{\text{def}}{=} \bigwedge_{\pi_i \in \Pi_i} (P_i(\boldsymbol{\pi}_i) = \tau_i(\pi_i))$$

Expressing Nash Equilibrium

Definition (Best response)

Given a mixed strategy profile σ, i's strategy is a best response if for every formula φ, we have

$$\sigma \models \varphi \to [(Ag \setminus \{i\})]\langle \sqsubseteq_i \rangle \varphi.$$

Given a specific σ we define

$$\mathrm{br}_i(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma} \to [(Ag \setminus \{i\})]\langle \sqsubseteq_i \rangle \boldsymbol{\sigma}.$$

A Nash equilibrium is a mixed strategy profile, such that everyone's strategy is a best response. For each σ, define

$$\mathrm{Nash}(\boldsymbol{\sigma}) \equiv \bigwedge_{i \in Ag} \mathrm{br}_i(\boldsymbol{\sigma}).$$

So σ is a Nash equilibrium in $\mathcal{G}$ if and only if $\models \mathrm{Nash}(\boldsymbol{\sigma})$

Axioms

- Basic axioms
- Global modality axioms
- Probability Bounds and Restrictions
- Inequality axioms

Axioms

- Basic axioms
  - Classical Logic Tautologies
  - $[\star](\varphi \to \psi) \to ([\star]\varphi \to [\star]\psi)$, with $\star \in \{G, \sqsubset_i, \sqsupset_i, =_i\}$.
  - $[*]\varphi \to \varphi$, with $* \in \{G, =_i\}$ (reflexivity)
  - $[i]\varphi \to [G]\varphi$ $(i \in G)$
  - $[\sqsubset_i]\varphi \to [=_i][\sqsubset_i]\varphi$ (neutrality of $=_i$)
  - $[\sqsupset_i]\varphi \to [=_i][\sqsupset_i]\varphi$
  - $\pm P_i(\boldsymbol{\pi}_i) \geq q \to [i] \pm P_i(\boldsymbol{\pi}_i) \geq q$ ($[i]$ fixes i's strategy)
  - $\pm \mathbf{u}_i \geq q \to [=_i] \pm \mathbf{u}_i \geq q$ ($[=_i]$ fixes i's utility)
  - $\mathbf{u}_i \geq q \to [\sqsubset_i]\mathbf{u}_i > q$ (strictness of $\sqsubset_i$)
  - $\mathbf{u}_i \leq q \to [\sqsupset_i]\mathbf{u}_i < q$
- Global modality axioms
- Probability Bounds and Restrictions
- Inequality axioms

Axioms

- Basic axioms
- Global modality axioms
  - $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\bigwedge_{i \in G} \boldsymbol{\sigma}_i \to \langle G \rangle \varphi)$
    (If φ is true at σ, then $\langle G \rangle \varphi$ is true in any τ that agrees with σ on the strategies of those in G.)
  - $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\mathbf{u}_i = u_i(\sigma) \to \langle =_i \rangle \varphi)$
  - $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\mathbf{u}_i < u_i(\sigma) \to \langle \sqsubset_i \rangle \varphi)$
  - $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\mathbf{u}_i > u_i(\sigma) \to \langle \sqsupset_i \rangle \varphi)$
- Probability Bounds and Restrictions
- Inequality axioms

Axioms

- Basic axioms
- Global modality axioms
- Probability Bounds and Restrictions
  - $P(\boldsymbol{\pi}) \geq 0$
  - $\sum_{\pi \in \Pi} P(\boldsymbol{\pi}) = 1$
  - $\neg \boldsymbol{\sigma}$ for each $\sigma \notin \widetilde{\Sigma}$ $\quad (\widetilde{\Sigma} \subseteq \Sigma)$
- Inequality axioms

Axioms

- Basic axioms
- Global modality axioms
- Probability Bounds and Restrictions
- Inequality axioms
  - (Permutation) $\sum_{k=1}^{n} q_k P(\pi_k) \geq q \rightarrow \sum_{k=1}^{n} q_{j_k} P(\pi_{j_k}) \geq q$
  - (Adding and deleting zero terms) $t \geq q \leftrightarrow t + 0P(\pi_{k+1}) \geq q$
  - (Adding coefficients) $\sum_{k=1}^{n} q_k P(\pi_k) \geq q \wedge \sum_{k=1}^{n} q'_k P(\pi_k) \geq q' \rightarrow \sum_{k=1}^{n} (q_k + q'_k) P(\pi_k) \geq (q + q')$
  - (Multiplying my a non-negative constant) $t \geq q \leftrightarrow dt \geq dq$ where $d > 0$
  - (Monotonicity) $(t \geq q) \rightarrow (t > q')$ where $q > q'$

Rules

Modus Ponens

$$\frac{A \vdash \varphi \to \psi \quad A \vdash \varphi}{A \vdash \psi}$$

Necessitation

$$\frac{\vdash \varphi}{\vdash [\star]\varphi} \qquad (\star \in \{G, =_i, \sqsubset_i, \sqsupset_i\})$$

Monotonicity

$$\frac{\vdash \varphi}{A \vdash \varphi}$$

Pseudo modalities and Existence rule

Pseudo-modalities: Let $s_i \in \{G, \sqsubset_i, \sqsupset_i, =_i\} \cup \mathcal{L}$. Define $[(s_1, \dots, s_n)]\varphi$ by

- $[()]\varphi \stackrel{\text{def}}{=} \varphi$
- $[(\psi, s_2, \dots, s_n)]\varphi \stackrel{\text{def}}{=} \psi \to [(s_2, \dots, s_n)]\varphi$
- $[(a, s_2, \dots, s_n)]\varphi \stackrel{\text{def}}{=} [a][(s_2, \dots, s_n)]\varphi$

Existence rule: For $s = (s_1, \dots, s_n)$

$$\frac{A \vdash [s](P(\pi) \neq q) \text{ for all } q \in \mathbb{Q}, q \neq p}{A \vdash [s](P(\pi) = p)}$$

A probability value *exists* for each pure profile and agent.

More infinitary rules

Guarded necessitation

$$\frac{\vdash \boldsymbol{\tau} \to \varphi \text{ for all } \tau \in \widetilde{\Sigma} \text{ such that } \tau_i = \sigma_i \text{ for all } i \in G}{\vdash \boldsymbol{\sigma} \to [G]\varphi}$$

(Compare with the axiom
$[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\bigwedge_{i \in G} \boldsymbol{\sigma}_i \to \langle G\rangle\varphi)$)

$$\frac{\vdash \boldsymbol{\tau} \to \varphi \text{ for all } \tau \in \widetilde{\Sigma} \text{ such that } u_i(\tau) = u_i(\sigma)}{\vdash \boldsymbol{\sigma} \to [=_i]\varphi}$$

$$\frac{\vdash \boldsymbol{\tau} \to \varphi \text{ for all } \tau \in \widetilde{\Sigma} \text{ such that } u_i(\tau) > u_i(\sigma)}{\vdash \boldsymbol{\sigma} \to [\sqsubset_i]\varphi}$$

$$\frac{\vdash \boldsymbol{\tau} \to \varphi \text{ for all } \tau \in \widetilde{\Sigma} \text{ such that } u_i(\tau) < u_i(\sigma)}{\vdash \boldsymbol{\sigma} \to [\sqsupset_i]\varphi}$$

Strong completeness

Theorem (Strong Completeness)

If $A \models \varphi$ *then* $A \vdash \varphi$

Nash equilibrium

Recall that

$$\mathsf{Nash}(\boldsymbol{\sigma}) \equiv \bigwedge_{i \in Ag} \boldsymbol{\sigma} \to [(Ag \setminus \{i\})]\langle \sqsubseteq_i \rangle \boldsymbol{\sigma}.$$

Theorem

$\widetilde{\Sigma}$ contains a Nash equilibrium if and only if for some (or all) $\tau \in \widetilde{\Sigma}$,

$$\{\neg[\emptyset] \mathit{Nash}(\boldsymbol{\sigma}) \mid \sigma \in \widetilde{\Sigma} \setminus \{\tau\}\} \vdash \mathit{Nash}(\boldsymbol{\tau}).$$

Remarks

- One game fixes both the language and the model
- Soundness and *strong* completeness is established
- Possible further developments might involve
  - dynamic removal of strategy profiles
  - non-trivial epistemics (agents may assume certain strategies of others or might be mistaken about their own strategy)
  - extensive game form
  - imperfect information