Modal logics and their semantics

Joshua Sack
Department of Mathematics and Statistics,
California State University Long Beach

California State University Dominguez Hills Feb 22, 2012

Relational structures

Definition

A finite *relational structure* is a pair (S, R), where

- S is a finite set
- R is a (binary) relation over S.

related notions

A relational structure is also known as a

- directed graph (graph theory)
- Kripke frame (modal logic)
- network (network theory)
- finite labelled transition system (computer science)
- automaton (computer science)

Logics that describe variations of graphs

Two basic logics are

First order logic

first order logic describes a graph using quantifiers and relations.

- can provide a complete description of a finite graph
- the truths of the logic are generally undecidable (the answer to the question "is ϕ true is *some* model/graph?" cannot necessarily be answered in a finite number of time-steps).

Modal logic

modal logic describes *properties* of vertices in a graph using locally defined quantifiers

- can only express bisimulation equivalence classes of a graph (highly relevant in modeling computation)
- the logic is generally decidable

What is modal logic?

Modal logic formulas φ are given by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi \mid \Diamond\varphi$$

where

- p is an "*atomic proposition*" coming from a fixed set Φ
- $\bot$ is a formula representing *falsity*
- $\neg$ is *negation*
- $\wedge$ is *conjunction*
- $\Box$ is *necessity*
- $\Diamond$ is *possibility*

$\vee$ (disjunction) and $\rightarrow$ (only if) are derived from $\neg$ and $\wedge$

Example

$\Diamond\Diamond p \rightarrow \Diamond p$ (related to transitivity)

Pointed models

Whereas first-order logic describes properties of a model, modal logic describes properties of a pointed model

Definition (Pointed Kripke model)

A *pointed Kripke model* over a set Φ is a tuple $((S, R, V), s)$, where

- (S, R) is a directed graph (or "Kripke frame")
- $V : \Phi \to \mathcal{P}(S)$ is a function assigning "atomic propositions" $p \in \Phi$ to subsets S.
- $s \in S$ is the "point"

The tuple (S, R, V) is called a *Kripke model*, and is a kind of directed vertex-labelled graph.

Semantics

Truth is defined by a relation $\models$ between pointed models and formulas as follows:

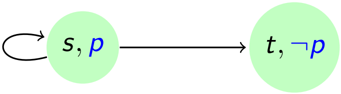
- $M, s \models p$ iff $s \in V(p)$
- $M, s \not\models \bot$ (where $\not\models$ is the complement of $\models$)
- $M, s \models \neg\varphi$ iff $M, s \not\models \varphi$
- $M, s \models \varphi \wedge \psi$ iff $M, s \models \varphi$ and $M, s \models \psi$
- $M, s \models \Box\varphi$ iff $M, t \models \varphi$ for all $t \in \{t \mid sRt\}$
- $M, s \models \Diamond\varphi$ iff $M, t \models \varphi$ for some $t \in \{t \mid sRt\}$.

Remark

Note: if $\{t \mid sRt\} = \emptyset$, then $M, s \models \Box\bot$, while $M, s \not\models \Diamond\bot$

Example

- $S = \{s, t\}$, $\Phi = \{p\}$



- $t \models \Box\bot$
- $s \models \Diamond p$
- $s \models \Diamond\neg p$
- $s \not\models \Box p$
- $s \models \Diamond\Box\bot$
- $s \not\models \Box\Box\bot$

Relationship to 1st order logic: Standard translation

Definition (Standard translation)

Let x be a first-order variable. The *standard translation* ST_x mapping modal to first order formulas is defined as follows

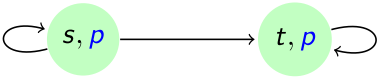
- $ST_x(p) = Px$ (a "predicate" P for each proposition letter p.)
- $ST_x(\bot) = x \neq x$ (a symbol for inequality in 1st order lang.)
- $ST_x(\neg\varphi) = \neg ST_x(\varphi)$
- $ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x\psi$
- $ST_x(\Box\varphi) = \forall y(Rxy \to ST_y\varphi)$
- $ST_x(\Diamond\varphi) = \exists y(Rxy \wedge ST_y\varphi)$

Proposition

For all M, s, and modal formulas φ,

$$M, s \models \varphi \text{ iff } M \models ST_x(\varphi)[s].$$

Expressivity and Invariants



For any modal formula, φ

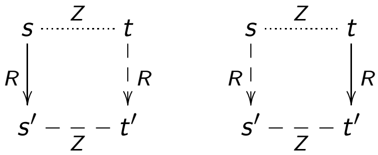
$$s \models \varphi \qquad \text{if and only if} \qquad t \models \varphi$$

Bisimulation

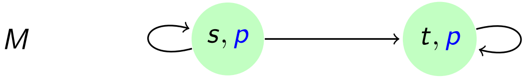
Definition

A bisimulation between models $M = (S^M, R^M, V^M)$ and $N = (S^N, R^N, V^N)$ is a relation $Z \subseteq S^M \times S^N$ obeying the constraint that if sZt, then

- $s \in V^M(p)$ if and only if $t \in V^N(p)$
- if $sR^M s'$ then there exists t', such that $tR^N t'$ and $s'Zt'$
- if $tR^N t'$ then there exists s', such that $sR^M s'$ and $s'Zt'$

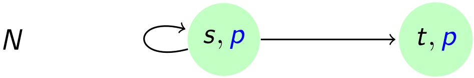


Bisimulation example



A bisimulation between M and itself: the total Cartesian product

$$Z = \{(s,s),(s,t),(t,s),(t,t)\}.$$



Largest bisimulation between N and itself:

$$Z = \{(s,s),(t,t)\}.$$

Largest bisimulation between M and N is $Z = \emptyset$.

Relationship to 1st order logic: Bisimulation

Let L be a first-order language with

- *inequality*
- *unary predicates* P from some set Φ, and
- a *relation symbol* R.

Theorem (Van Benthem Characterization Theorem)

A first-order formula α in L is invariant under bisimulation if and only if $\alpha = ST_x(\varphi)$ for some modal formula φ.

in other words, modal logic is the bisimulation invariant fragment of 1st order logic.

Frame semantics

Definition (frame validity)

- A formula φ is *valid* in a pointed frame $((S, R), s)$, written $(S, R), s \models \varphi$ if and only if for all valuations V, $(S, R, V), s \models \varphi$.
- A formula φ is *valid* in a frame (S, R), written $(S, R) \models \varphi$ if for all $s \in S$, $(S, R), s \models \varphi$.

Observation

Modal logic with frame validity is a fragment monadic second order logic.

This is because we quantify over valuations (subsets of the model).

- $F, s \models \varphi$ iff $F \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[s]$
- $F \models \varphi$ iff $F \models \forall P_1 \ldots \forall P_n \forall x ST_x(\varphi)$

Definability

Definition (frame definability)

A modal formula φ *defines* a class of frames $\mathcal{F}$ if

$$\mathcal{F} = \{F \mid F \models \varphi\}.$$

Examples

- $\mathcal{F} \models p \to \Diamond p$ iff $\mathcal{F} \models \forall x Rxx$ (reflexivity)
- $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$ iff $\mathcal{F} \models \forall x \forall y \forall z (Rxy \wedge Ryz \to Rxz)$ (transitivity)
- $\mathcal{F} \models \Diamond p \to \Box\Diamond p$ iff $\mathcal{F} \models \forall x \forall y \forall z (Rxy \wedge Rxz \to Ryz)$ (Euclidean)

But here is a modal formula that defines a class of frames that cannot be defined by a first-order logic formula:

- $\Box(\Box p \to p) \to \Box p$ characterizes all frames where R is transitive and converse well-founded (there is no infinite path emanating from a point).

Goldblatt-Thomason Theorem

Theorem

A first order definable class of frames $\mathcal{F}$ is modally definable if and only if it is closed under each of the following:

- *bounded morphic images*
- *generated subframes*
- *disjoint unions*
- *ultrafilter extensions.*

Definition (bounded morphic image)

A bounded morphism from (S, R) to (S', R') is a function $f : S \to S'$, such that both of the following hold:

- sRt implies $f(s)R'f(t)$
- if $f(s)R't'$, then there exists t, such that sRt and $t' = f(t)$.

Goldblatt-Thomason Theorem

Theorem

A first order definable class of frames $\mathcal{F}$ is modally definable if and only if it is closed under each of the following:

- *bounded morphic images*
- *generated subframes*
- *disjoint unions*
- *ultrafilter extensions.*

Definition (bounded morphic image)

A bounded morphism from (S, R) to (S', R') is a function $f : S \to S'$, such that both of the following hold:

- sRt implies $f(s)R'f(t)$
- if $f(s)R't'$, then there exists t, such that sRt and $t' = f(t)$.

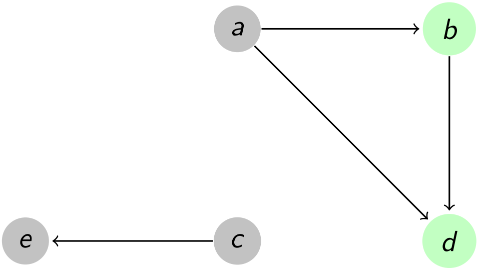
Generated submodel

Definition (Generated submodel)

(S, R) is a generated subframe of (S', R') if

- $S \subseteq S'$,
- $R \subseteq R'$, and
- $(s \in S$ and $sR't)$ implies $(t \in S$ and $sRt)$

Example



Ultrafilter extensions

Definition (Ultrafilter)

An ultrafilter is a set $\mathcal{F}$ of sets in S, such that $\emptyset \notin \mathcal{F}$ and

- $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$ (closed under superset)
- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ (closed under finite intersection)
- $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ (either a set or its complement is in $\mathcal{F}$.

Definition (Ultrafilter extension)

Given a frame (S, R) its ultrafilter extension is the frame (S', R'), where

- S' is the set of ultrafilters on S
- $\mathcal{F} R' \mathcal{F}'$ if and only if $\{s \mid \exists s',\ sRs',\ s' \in X\} \in \mathcal{F}$ whenever $X \in \mathcal{F}'$.

Expressing the out-degree

Graded modal logic is given by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_n\varphi \mid \Diamond_n\varphi$$

- $M, s \models \Box_n\varphi$ if and only if
  for all distinct $t_1, \ldots, t_n \in \{t \mid sRt\}$
  for some i, $M, t_i \models \varphi$.
- $M, s \models \Diamond_n\varphi$ if and only if
  for some distinct $t_1, \ldots, t_n \in \{t \mid sRt\}$
  for all i, $M, t_i \models \varphi$.

Note that

- $\neg\Box_n\neg\varphi$ is equivalent to $\Diamond_n\varphi$ ($\Box_n, \Diamond_n$ are duals)
- $\Box_1\varphi$ is equivalent to $\Box\varphi$
- $\Diamond_1\varphi$ is equivalent to $\Diamond\varphi$

"The out-degree is n" can be expressed as $\Diamond_n\neg\bot \wedge \Box_{n+1}\bot$.

Expressing the out-degree

Graded modal logic is given by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_n \varphi \mid \Diamond_n \varphi$$

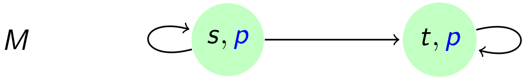
- $M, s \models \Box_n \varphi$ if and only if
  for all distinct $t_1, \ldots, t_n \in \{t \mid sRt\}$
  for some i, $M, t_i \models \varphi$.
- $M, s \models \Diamond_n \varphi$ if and only if
  for some distinct $t_1, \ldots, t_n \in \{t \mid sRt\}$
  for all i, $M, t_i \models \varphi$.

Note that

- $\neg\Box_n\neg\varphi$ is equivalent to $\Diamond_n\varphi$ ($\Box_n, \Diamond_n$ are duals)
- $\Box_1\varphi$ is equivalent to $\Box\varphi$
- $\Diamond_1\varphi$ is equivalent to $\Diamond\varphi$

"The out-degree is n" can be expressed as $\Diamond_n\neg\bot \wedge \Box_{n+1}\bot$.

Example



The formula $\Box_2\bot$ is false at s but true at t.

Additional property of graded modal logic

- Satisfiability of graded modal logic is decidable,
- The complexity of the satisfiability problem is PSPACE-complete (as is ordinary modal logic).

Weighted edges

Weighted modal logic is given by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_a\varphi \mid \Diamond_a\varphi$$

Replace R with directed weighted edges R_q with $q \in \mathbb{Q}$.

- For $t \in S$, let $P_s(t) = q$ where sR_qt
- For $T \subseteq S$, let $P_s(T) = \sum_{\{t\in T \mid \exists q.\ sR_qt\}} P_s(t)$.

Then

- $M, s \models \Box_a\varphi$ if and only if
  for all $T \subseteq S$, such that $P_s(T) \geq a$,
  for some $t \in T$, $M, t \models \varphi$.
- $M, s \models \Diamond_a\varphi$ if and only if
  for some $T \subseteq S$, such that $P_s(T) \geq a$,
  for all $t \in T$, $M, t \models \varphi$.

If $q = 1$ always, then this is graded modal logic.
If $P_s(S) = 1$ always, then this is modal probability logic:
$\Diamond_a\varphi$ is read "the probability of φ is at least a"

Weighted edges

Weighted modal logic is given by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_a \varphi \mid \Diamond_a \varphi$$

Replace R with directed weighted edges R_q with $q \in \mathbb{Q}$.

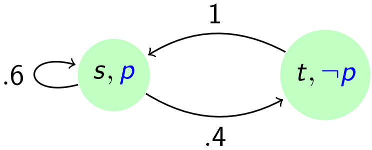
- For $t \in S$, let $P_s(t) = q$ where $sR_q t$
- For $T \subseteq S$, let $P_s(T) = \sum_{\{t \in T \mid \exists q.\ sR_q t\}} P_s(t)$.

Then

- $M, s \models \Box_a \varphi$ if and only if
  for all $T \subseteq S$, such that $P_s(T) \geq a$,
  for some $t \in T$, $M, t \models \varphi$.
- $M, s \models \Diamond_a \varphi$ if and only if
  for some $T \subseteq S$, such that $P_s(T) \geq a$,
  for all $t \in T$, $M, t \models \varphi$.

If $q = 1$ always, then this is graded modal logic.
If $P_s(S) = 1$ always, then this is modal probability logic:
$\Diamond_a \varphi$ is read "the probability of φ is at least a"

Probability logic example



$\Diamond_a \varphi$ is read "the probability of (transitioning to a state where) φ is at least a."
$\Box_a \varphi$ is read "the probability of (transitioning to a state where) $\neg\varphi$ is less than a."

- $s \models \Diamond_{.4} p \wedge \Diamond_{.4} \neg p$
- $s \not\models \Diamond_{.6} \neg p$
- $s \models \Diamond_{.4} \Diamond_1 p$ (higher-order probabilities)

Linear combinations in probability logic

Additivity conditions are prevalent in probability: some probability logics make linear combinations explicit as “terms” t

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid t \geq a$$

$$t ::= aP(\varphi) \mid t_1 + t_2$$

Here we can express *additivity*:

$$P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) - P(\varphi) = 0$$

where $t = 0$ is an abbreviation for $(t \geq 0) \wedge (-t \geq 0)$.

Linear combinations in probability logic

Additivity conditions are prevalent in probability: some probability logics make linear combinations explicit as “terms” t

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid t \geq a$$

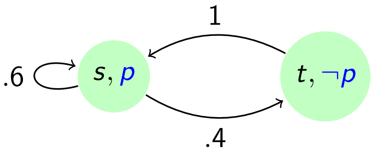
$$t ::= aP(\varphi) \mid t_1 + t_2$$

Here we can express *additivity*:

$$P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) - P(\varphi) = 0$$

where $t = 0$ is an abbreviation for $(t \geq 0) \wedge (-t \geq 0)$.

Probability logic with linear combinations example



- $s \models P(p) \geq .4 \wedge P(\neg p) \geq .4$
- $s \not\models P(p) \geq .6$
- $s \models P(P(p) \geq 1) \geq .4$ (higher-order probabilities)
- $s \models P(p) - P(\neg p) \geq .2$
- $s \models -P(p) + P(\neg p) \geq -.2$

The last two examples imply that $P(p) - P(\neg p) = .2$.

Decidability

- Satisfiability of modal probability logic is decidable,
- The complexity of the satisfiability problem is PSPACE-complete (as is ordinary and graded modal logics).

Topological semantics

Replace the relation with a topology $\mathcal{T}$. Topological model: $(S, \mathcal{T}, V)$

- $M, s \models \Box\varphi$ if and only if

  for some $O \in \mathcal{T}$, such that $s \in O$
  for all $t \in O$, $M, t \models \varphi$

  $\Box\varphi$ is true in the interior of the set $\{t \mid t \models \varphi\}$

- $M, s \models \Diamond\varphi$ if and only if

  for all $O \in \mathcal{T}$, such that $s \in O$
  for some $t \in O$, $M, t \models \varphi$.

  $\Diamond\varphi$ is true in the closure of the set $\{t \mid t \models \varphi\}$

Remark

Note that the quantifier pattern here is the opposite of those used for weighted modal logic (including graded modal logic and modal probability logic).

Topological semantics

Replace the relation with a topology $\mathcal{T}$. Topological model: $(S, \mathcal{T}, V)$

- $M, s \models \Box\varphi$ if and only if

  for some $O \in \mathcal{T}$, such that $s \in O$

  for all $t \in O$, $M, t \models \varphi$

  $\Box\varphi$ is true in the interior of the set $\{t \mid t \models \varphi\}$

- $M, s \models \Diamond\varphi$ if and only if

  for all $O \in \mathcal{T}$, such that $s \in O$

  for some $t \in O$, $M, t \models \varphi$.

  $\Diamond\varphi$ is true in the closure of the set $\{t \mid t \models \varphi\}$

Remark

Note that the quantifier pattern here is the opposite of those used for weighted modal logic (including graded modal logic and modal probability logic).

Neighborhood semantics

Let a neighborhood model be $M = (S, R, V)$ where $R \subseteq S \times \mathcal{P}(S)$. Let $N_s = \{T \subseteq S \mid sRT\}$.

- $M, s \models \Box\varphi$ if and only if $\{t \in S \mid M, t \models \varphi\} \in N_s$.
- $M, s \models \Diamond\varphi$ if and only if $\{t \in S \mid M, t \not\models \varphi\} \notin N_s$.

If N_s is upward closed ($T \subseteq T'$ and $T \in N_s$ implies $T' \in N_s$), then

- $M, s \models \Box\varphi$ if and only if
  for some $T \in N_s$
  for all $t \in T$, $M, t \models \varphi$.
- $M, s \models \Diamond\varphi$ if and only if
  for all $T \in N_s$
  for some $t \in T$, $M, t \models \varphi$.

Remark

In the case of topological semantics, N_s includes every set with s in its interior.

Neighborhood semantics

Let a neighborhood model be $M = (S, R, V)$ where $R \subseteq S \times \mathcal{P}(S)$. Let $N_s = \{T \subseteq S \mid sRT\}$.

- $M, s \models \Box\varphi$ if and only if $\{t \in S \mid M, t \models \varphi\} \in N_s$.
- $M, s \models \Diamond\varphi$ if and only if $\{t \in S \mid M, t \not\models \varphi\} \notin N_s$.

If N_s is upward closed ($T \subseteq T'$ and $T \in N_s$ implies $T' \in N_s$), then

- $M, s \models \Box\varphi$ if and only if
  for some $T \in N_s$
  for all $t \in T$, $M, t \models \varphi$.
- $M, s \models \Diamond\varphi$ if and only if
  for all $T \in N_s$
  for some $t \in T$, $M, t \models \varphi$.

Remark

In the case of topological semantics, N_s includes every set with s in its interior.

Neighborhood semantics

Let a neighborhood model be $M = (S, R, V)$ where $R \subseteq S \times \mathcal{P}(S)$. Let $N_s = \{T \subseteq S \mid sRT\}$.

- $M, s \models \Box\varphi$ if and only if $\{t \in S \mid M, t \models \varphi\} \in N_s$.
- $M, s \models \Diamond\varphi$ if and only if $\{t \in S \mid M, t \not\models \varphi\} \notin N_s$.

If N_s is upward closed ($T \subseteq T'$ and $T \in N_s$ implies $T' \in N_s$), then

- $M, s \models \Box\varphi$ if and only if
  for some $T \in N_s$
  for all $t \in T$, $M, t \models \varphi$.
- $M, s \models \Diamond\varphi$ if and only if
  for all $T \in N_s$
  for some $t \in T$, $M, t \models \varphi$.

Remark

In the case of topological semantics, N_s includes every set with s in its interior.

Generalizations

Generalization of weighted modal logic

If N_s consists of all sets whose weighted edges from s add to at least a, then $\Box\varphi$ in the neighborhood semantics is equivalent to $\Diamond_a\varphi$ in the weighed modal logic semantics.

- Recall that weighed modal logic generalizes graded modal logic and modal probability logic.
- Recall furthermore that graded modal logic generalizes standard relational modal logic.

Generalization of standard relational modal logic

If N_s consists of all supersets of the the $\{t \mid sRt\}$, then $\Box\varphi$ is the same in both the neighborhood semantics and in the standard relational model semantics.

Generalizations

Generalization of weighted modal logic

If N_s consists of all sets whose weighted edges from s add to at least a, then $\Box\varphi$ in the neighborhood semantics is equivalent to $\Diamond_a\varphi$ in the weighed modal logic semantics.

- Recall that weighed modal logic generalizes graded modal logic and modal probability logic.
- Recall furthermore that graded modal logic generalizes standard relational modal logic.

Generalization of standard relational modal logic

If N_s consists of all supersets of the the $\{t \mid sRt\}$, then $\Box\varphi$ is the same in both the neighborhood semantics and in the standard relational model semantics.

THANK YOU!