The many classical faces of quantum structures

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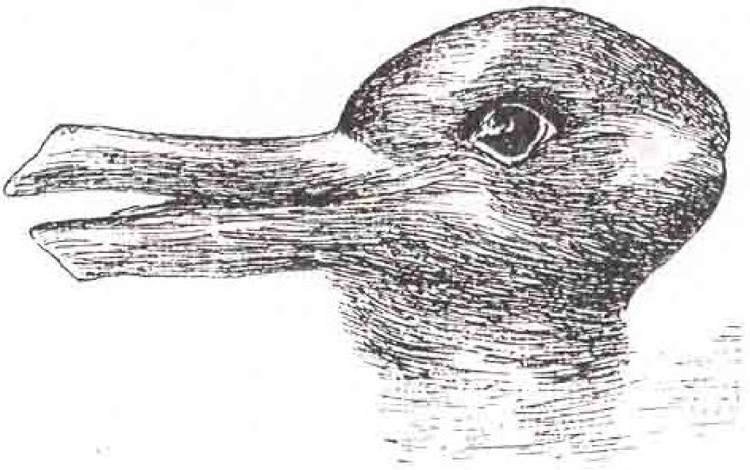
Classical faces of quantum structures

I Introduction
II Order theory
III Operator algebra
IV Interaction

Relationship between classical and quantum



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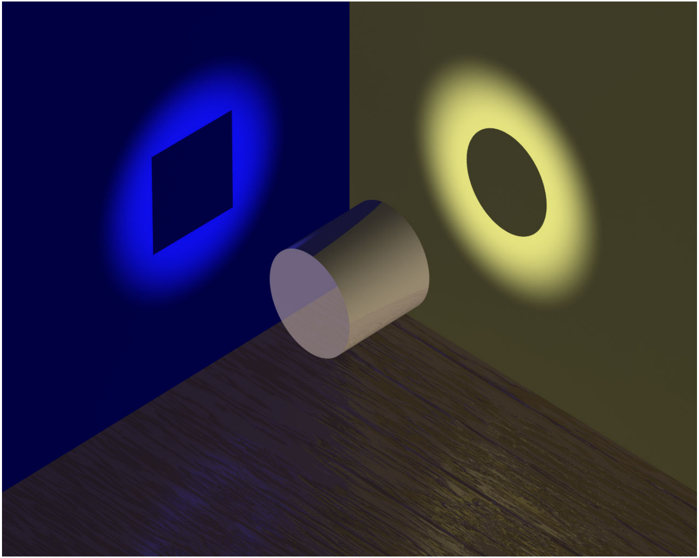

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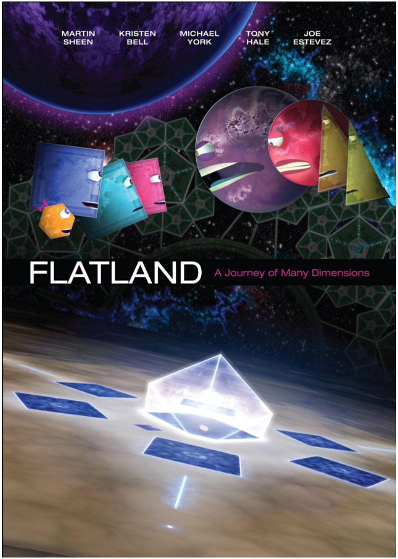

(In)compatibilities between measurements

- State space = Hilbert space
  Sharp measurements = projection-valued measures
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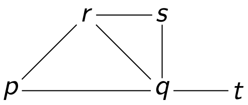
- (In)compatibilities form graph:



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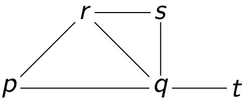

-    **Theorem**: Any graph can be realised as PVMs on a Hilbert space.

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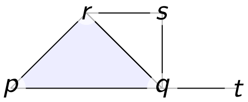

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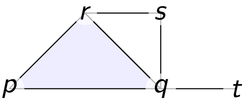
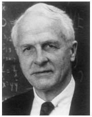
- (In)compatibilities now form abstract simplicial complex:



-    **Theorem**: Any abstract simplicial complex can be realised as POVMs on a Hilbert space.

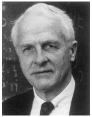
Quantum logic

~~Subsets of a set~~

Subspaces of a Hilbert space

Quantum logic

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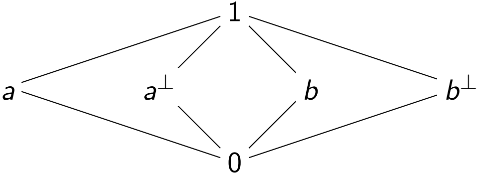
orthomodular lattice



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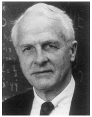 

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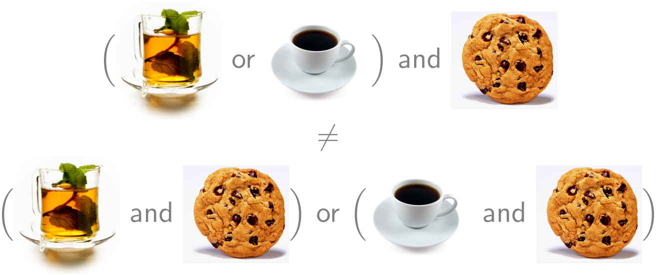

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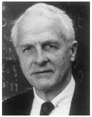


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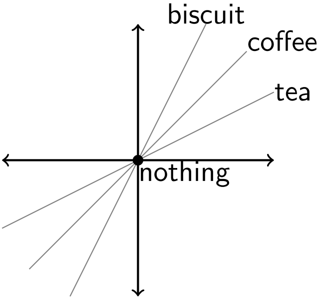

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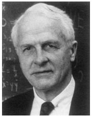


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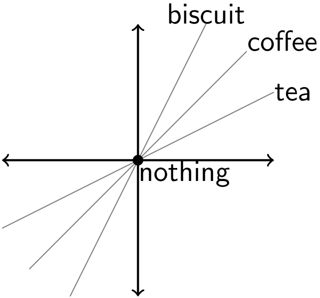
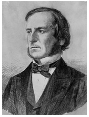
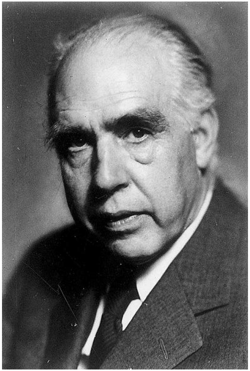




However: fine when within orthogonal basis

(Boolean subalgebra)

Doctrine of classical concepts



"However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms.... The argument is simply that by the word experiment we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangements and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics."

Kochen–Specker

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  (state $\alpha|0\rangle + \beta|1\rangle$ gives outcome 0 with probability $|\alpha|^2$)

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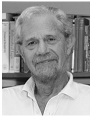
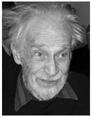
- A *hidden variable* for a state is an assignment of a consistent outcome to any possible measurement.

Theorem: hidden variables cannot exist
(if dimension ≥ 3.)

Part I

Order theory

Piecewise structures

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Piecewise structures

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- A piecewise Boolean algebra is a set B with:
  - a reflexive binary relation $\odot \subseteq B^2$;
  - (partial) binary operations $\vee, \wedge \colon \odot \to B$;
  - a (total) function $\neg \colon B \to B$;

  such that every $S \subseteq B$ with $S^2 \subseteq \odot$ is contained in a $T \subseteq B$ with $T^2 \subseteq \odot$ where $(T, \wedge, \vee, \neg)$ is a Boolean algebra.

Piecewise structures

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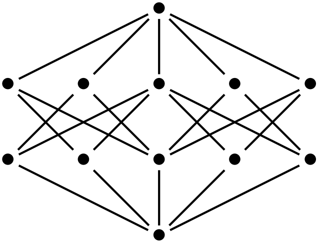

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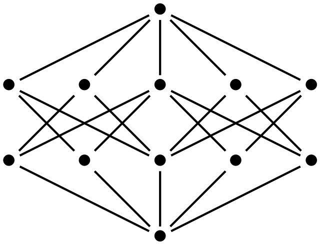
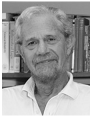
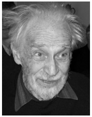
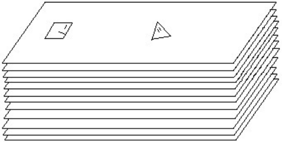


-   **Theorem**: There is no piecewise morphism $\mathrm{Proj}(\mathbb{C}^3) \to \{0, 1\}$

Classical viewpoints

- Given a piecewise Boolean algebra P,
  consider $\mathcal{C}(P) = \{B \subseteq P \text{ Boolean subalgebra}\}$,
  the collection of classical viewpoints.



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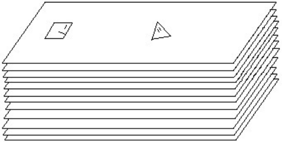

-   **Theorem**: Can reconstruct P
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  $(P \cong \operatorname{colim} \mathcal{C}(P))$

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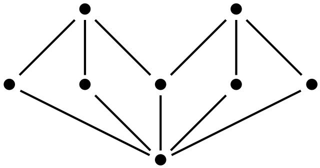

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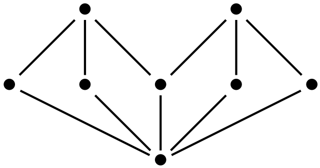
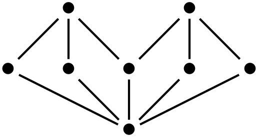


- 

  Theorem: $\mathcal{C}(P)$ determines P
  $(P \cong P' \iff \mathcal{C}(P) \cong \mathcal{C}(P'))$
  shape of parts determines whole

Piecewise Boolean algebras



Theorem: If a poset L:

- has directed suprema;
- has nonempty infima;
- each element is a supremum of compact ones;
- each downset is cogeometric with a modular atom;
- each element of height $n \leq 3$ covers $\binom{n+1}{2}$ elements;

then $L \cong \mathcal{C}(P)$ for a piecewise Boolean algebra P;
"L is a spectral poset".

Piecewise Boolean algebras

Lemma: If L is a spectral poset, there is a functor $F \colon L \to \mathbf{Bool}$ that preserves subobjects; "F is a spectral diagram".
$(\mathcal{C}(F(x)) \cong {\downarrow}x$, and $P = \operatorname{colim} F)$

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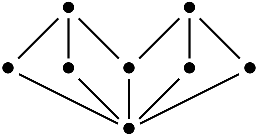

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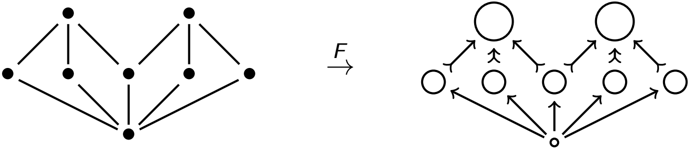

Piecewise Boolean algebras

Theorem: The following categories are equivalent:

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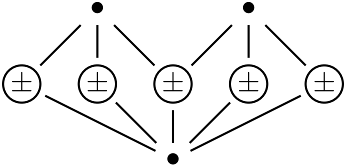

Part II

Operator algebra

Algebras of observables

Observables are primitive, states are derived

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C*-algebras

$*$-algebra of operators that is closed

AW*-algebras

abstract/algebraic version of W*-algebra

von Neumann algebras / W*-algebras

$*$-algebra of operators that is weakly closed

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∗-algebra of operators that is closed

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abstract/algebraic version of W*-algebra

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Jordan algebras
JC/JW-algebras: real version of above

Classical mechanics

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-  **Theorem**: Every commutative operator algebra is of this form.

- Can recover states (as maps $C(X) \to \mathbb{C}$): "spectrum"
  Constructions on states transfer to observables:

$$X + Y \mapsto C(X) \otimes C(Y)$$
$$X \times Y \mapsto C(X) \oplus C(Y$$

Equivalence of categories: states determine everything

Quantum mechanics

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- Recover states?
  Do states determine everything?
  "Noncommutative spectrum"?

Quantum state spaces?

certain convex sets (states)

sheaves over locales (prime ideals)

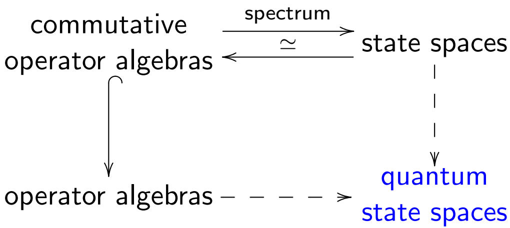
quantales (maximal ideals)

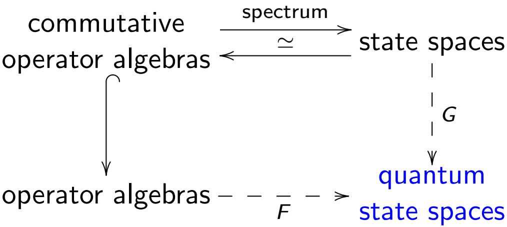
orthomodular lattices (projections)

q-spaces (projections of enveloping W*-algebra)

Quantum state spaces?



Quantum state spaces? No!



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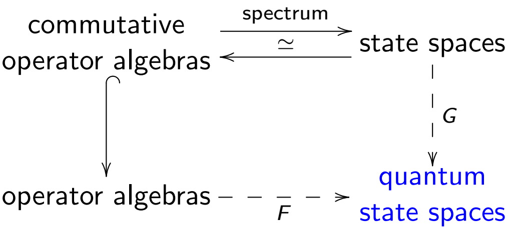

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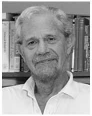
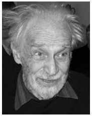
-   That's right. $(F(\mathbb{M}_n) = \emptyset$ for $n \geq 3.)$

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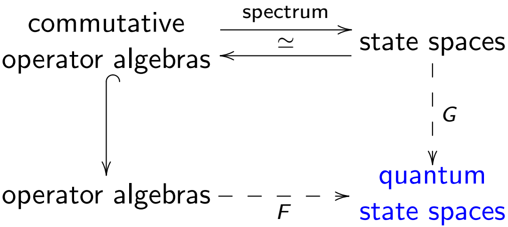

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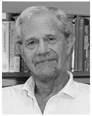
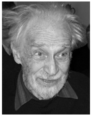
-   That's right. ($F(\mathbb{M}_n) = \emptyset$ for $n \geq 3$.)
- So G better not be continuous

  So quantum state spaces must be radically different ...

Classical viewpoints again

- Invariant that circumvents the obstruction:
  Given an operator algebra A,
  consider $\mathcal{C}(A) = \{C \subseteq A \text{ commutative subalgebra}\}$,
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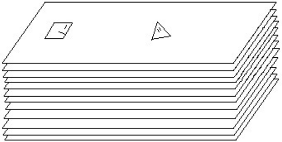

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  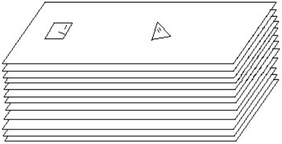

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- How much is this? Quite a bit:
  - Quantum foundations: Bohrification
  - Quantum logic: Bohrification
  - Quantum information theory: entropy

Contextual entropy


Define: contextual entropy of state ρ of A

function $E_\rho \colon \mathcal{C}(A) \to \mathbb{R}$,

$C \mapsto$ Shannon entropy $H(\mathrm{tr}(\rho\, -))$

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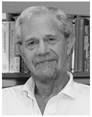
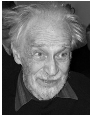
Theorem: E_ρ determines ρ!
(in dim ≥ 3)

Bohrification: history

  reformulate   with classical viewpoints

  general topos approach to physics

   Bohrification

  attempts at dynamics

Bohrification: idea

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  assignment of set $S(C)$ to each classical viewpoint $C \in \mathcal{C}(A)$
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-    **Theorem**: $\mathcal{T}(A)$ believes that $\underline{A}$ is a *commutative* operator algebra!

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Change rules to make quantum system classical. Price:

- No proof by contradiction. ($P \vee \neg P$)
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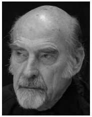 

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Theorem: $\underline{A}$ determined by state space (within $\mathcal{T}(A)$)

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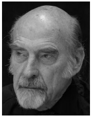  No matter!

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Circumvents obstruction ...

Piecewise structures: how far can we get?

 

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- So need to add more information to $\mathcal{C}(A)$ ...

Part III

Interaction between classical viewpoints

Five stages of grief

Established psychology:

Five stages of grief

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1. **Denial:** "These are not groups!"

Five stages of grief

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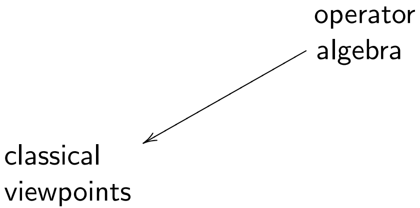
5. Acceptance: "Noncommutative groups are cool!"
6. Stockholm syndrome: "Commutative groups? Don't care!"

Active lattices: idea



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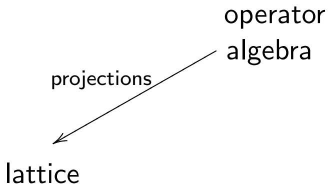

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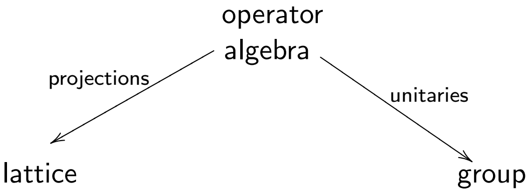

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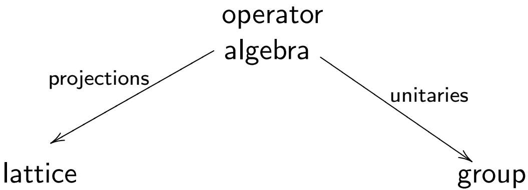

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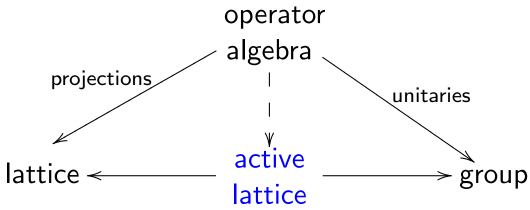

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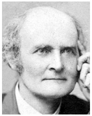
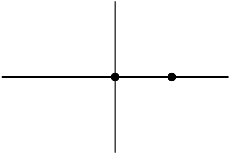
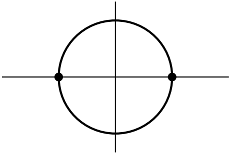
  So projections act on themselves!

Symmetries

-   $\longmapsto$ 

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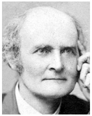
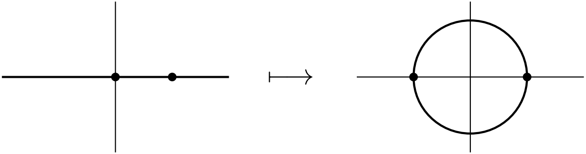
-  

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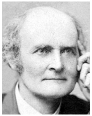
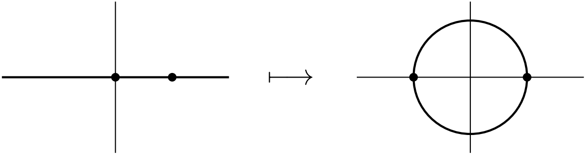
-  

  $p \longmapsto 1 - 2p$

  projections $\cong$ self-adjoint unitaries = "symmetries"

- Sym(A) is subgroup of unitaries generated by symmetries
- if A type I_1, then $\text{Sym}(A) = \{ \text{ all symmetries } \}$
- if A type $I_2/I_3/\ldots$, then $\text{Sym}(A) = \{ u \mid \det(u)^2 = 1 \}$
- if A type $I_\infty/II/III$, then $\text{Sym}(A) = \{ \text{ all unitaries } \}$

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  - an action of G on P

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-   **Theorem**: Its active lattice determines A (full and faithful functor)

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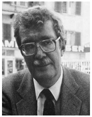
-    **Theorem**: Classical viewpoints in $\mathbb{M}_n(A)$ are diagonal.

  $(\forall C \in \mathcal{C}(\mathbb{M}_n(A)) \, \exists u \in U(\mathbb{M}_n(A)) \colon uCu^* \text{ diagonal})$

Matrix algebras: projections

-  Even if A has few projections, $\mathbb{M}_n(A)$ has lots!

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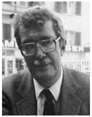
- 
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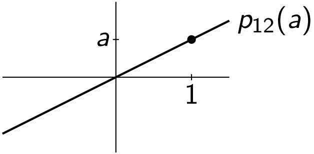

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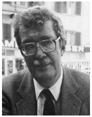
- 
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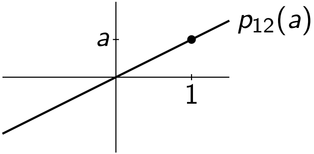

- These vector projections encode algebraic structure of A!
  $p_{ij}(a+b) =$ polynomial in $p_{ij}(a), p_{ik}(b), p_{jk}(c), \ldots$
  $p_{ij}(ab) =$ polynomial in $p_{ik}(a), p_{kj}(b), \ldots$
  $p_{ij}(a^*) =$ polynomial in $p_{ji}(a), \ldots$

Active lattices determine operator algebras

-   **Theorem**: Its active lattice determines A
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Active lattices determine operator algebras

- **Lemma**: If $f\colon \mathrm{Proj}(\mathbb{M}_n(A)) \to \mathrm{Proj}(\mathbb{M}_n(B))$ equivariant,
  then $f(p_{ij}(a)) = p_{ij}(\varphi(a))$ for some $\varphi\colon A \to B$.

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- What is the logic of such things ... ??

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"Knowing a quantum system $=$
all classical viewpoints $+$ switching between them"

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- Noncommutative topology, database theory, computability ...

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Part V

Bonus: abstract nonsense

Abstract quantum logic

- Topos logic not operational since “set-based”: propositions are “contextual subsets”
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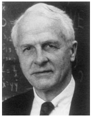
-   Propositions are closed subspaces
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Abstract quantum logic

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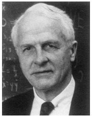
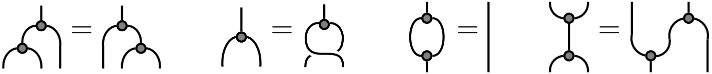
-   Propositions are closed subspaces (orthomodular projection lattice)
-   Quantum logic in abstract categories including "modal" quantifier $\exists$ ("dagger kernel categories" like **Hilb** or **Rel**)

Abstract operator algebras

- An abstract operator algebra (Frobenius algebra) in a tensor category is a morphism : $A \otimes A \to A$ satisfying



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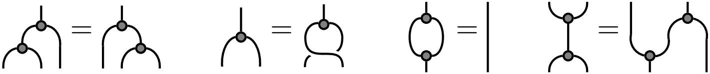

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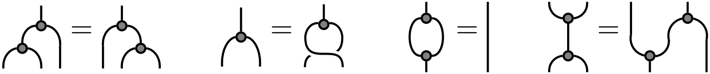

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Possibilistic quantum logic

-   Abstract quantum logic in **Rel** is classical modal logic
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- Can we reconstruct an abstract operator algebra from its *category* of classical viewpoints?

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