A Royal Road to Quantum Theory (or thereabouts)

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Outline

I Recent (and older) reconstructions of QL

II (Generalized) pobability Theory

III Conjugates and filters

IV Categories of Jordan models

I. Reconstructing QM

Ordered vector spaces

An **ordered vector space** is a real vector space $\mathbf{E}$ (for our purposes, finite-dimensional) with a designated **positive cone** $K \subseteq \mathbf{E}$:

- $a, b \in K \;\Rightarrow\; a + b \in K$;
- $a \in K \;\Rightarrow\; ta \in K$ for all $t \in \mathbb{R}_+$;
- $K \cap -K = \{0\}$;
- $\text{span}(K) = K - K = \mathbf{E}$.

Define $a \leq b$ iff $b - a \in K$ (so $a \geq 0$ iff $a \in K$).

Notation: $K = \mathbf{E}_+$.

Quantum Probability in a Nutshell

$\mathcal{H}$ a finite-dimensional complex Hilbert space; $\mathbf{E}(\mathcal{H}) :=$ space of hermitian operators on $\mathcal{H}$, ordered by the cone $\mathbf{E}_+(\mathcal{H})$ of positive operators.

- *observables* with values in a set $S \leftrightarrow$ mapping $a : S \to \mathbf{E}(\mathcal{H})_+$ with $\sum_{s \in S} a(s) = 1$;
- states $\leftrightarrow$ positive normalized linear functional α on $\mathbf{E}(\mathcal{H})$.
- *composite systems*: Given systems represented by $\mathcal{H}_1$, $\mathcal{H}_2$, composite system corresponds to $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- Also allowed: direct sums, e.g., $\mathbf{E}(\mathcal{H}_1) \oplus \mathbf{E}(\mathcal{H}_2)$. etc.

Nature takes this seriously — it works!

...But WHY?

Can we motivate this structure?

A Strategy:

1. Start with a very general (and conceptually simple) version of probability theory
2. Identify quantum probability theory as a special case
3. Add (simple?) constraints in hopes of singling out QM

Success will depend on one's view of the plausibility and simplicity of the constraints. (This is part of the fun!)

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- And lots more!
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This century: lots of new work, mainly from the QIT community (Hardy (2000), Dakič-Brukner(2008), Masanes-Mueller(2010), CDP (2010), ...)

- Focus on *finite-dimensional* QM + properties of *composite* (entangled) systems
- Weaker (and less *ad hoc*) axioms $\Rightarrow$ *finite-dimensional* QM.

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(2) Even with these assumptions, derivation of QM still seems rather involved.

Widening the target

(Baez, 2012): Real/quaternionic Hilbert space = pair $(\mathcal{H}, J)$: $\mathcal{H}$ a complex Hilbert space, J anti-unitary with $J^2 = +\mathbf{1}$ ($\mathbb{R}$) or $-\mathbf{1}$ ($\mathbb{H}$). Set

$$(\mathcal{H}_1, J_1) \otimes (\mathcal{H}_2, J_2) = (\mathcal{H}_1 \otimes \mathcal{H}_2, J_1 \otimes J_2) :$$

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Goal: a simple axiomatic framework allowing for finite-dimensional $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{H}$-QM (and not too much more) — ideally, without working too hard.

Jordan Algebras

This was *almost* done in 1934!

A euclidean Jordan algebra is a finite-dimensional real inner product space $\mathbf{E}$ with a commutative bilinear product $x, y \mapsto x \cdot y$ satisfying

- $x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y)$;
- $\exists u \in \mathbf{E}$, $u \cdot x = x \; \forall x \in \mathbf{E}$;
- $\langle x \cdot y, z \rangle = \langle y, x \cdot y \rangle$

Euclidean Jordan Algebras

Theorem [Jordan, von Neumann, Wigner, 1934] All euclidean Jordan algebras are direct sums of the following types:

- Hermitian parts of real, complex, quaternionic matrix algebras: $H_n(\mathbb{R})$, $H_n(\mathbb{C})$, $H_n(\mathbb{H})$, with $a \cdot b = \frac{1}{2}(ab + ba)$
- Spin Factors: $V_n =$ the euclidean space $\mathbb{R} \times \mathbb{R}^n$, with Jordan product
$$(t, \mathbf{x}) \cdot (s, \mathbf{y}) = (ts + \langle \mathbf{x}, \mathbf{y} \rangle, t\mathbf{y} + s\mathbf{x}).$$
(Note: can be embedded in $H_{2^n}(\mathbb{C})$).
- The *exceptional Jordan algebra* $H_3(\mathbb{O})$; $a \cdot b = \frac{1}{2}(ab + ba)$.

The Koecher-Vinberg Theorem

The Jordan product is hard to interpret. Fortunately, we don't need it!

Any EJA $\mathbf{E}$ is also an ordered real vector space with cone $\mathbf{E}_+ = \{a^2 | a \in \mathbf{E}\}$. An ordered space $\mathbf{E}$ is

- self-dual with respect to an inner product $\langle , \rangle$ iff $a \in \mathbf{E}_+$ iff $\langle a, b \rangle \geq 0$ for all $b \in \mathbf{E}_+$.
- homogeneous if the group of order-automorphisms of $\mathbf{E}$ acts transitively on the *interior* of the positive cone $\mathbf{E}_+$.

Theorem [Koecher, 1958; Vinberg, 1961]
Euclidean Jordan Algebras $\Leftrightarrow$ *HSD ordered vector spaces.*

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Theorem [Koecher, 1958; Vinberg, 1961]
Euclidean Jordan Algebras $\Leftrightarrow$ *HSD ordered vector spaces.*

Goal (revised): Simple axioms leading to a representation of physical systems in terms of HSD ordered vector spaces.

II. (General) Probability Theory

Probabilistic models

A test space is a pair $(X, \mathcal{M})$: X a space of *outcomes*, $\mathcal{M}$ a covering of X by non-empty finite sets called tests, representing outcome-sets of various possible measurements, experiments, etc.

A probability weight on $(X, \mathcal{M})$: a function $\alpha : X \to [0, 1]$ with $\sum_{x \in E} \alpha(x) = 1$ for every $E \in \mathcal{M}$.

A probabilistic model is a structure $A = (X, \mathcal{M}, \Omega)$:

- $(X, \mathcal{M})$ a test space,
- Ω a closed, convex set of probability weights on $\mathcal{M}$.

Notation: $\mathcal{M}(A)$, $X(A)$ and $\Omega(A)$ ...

The spaces $\mathbf{E}(A)$ and $\mathbf{V}(A)$

A probabilistic model A generates a pair of ordered vector spaces:

- $\mathbf{V}(A)$ = span of $\Omega(A)$ in $\mathbb{R}^{X(A)}$, with positive cone

$$\mathbf{V}(A)_+ := \{t\alpha | \alpha \in \Omega, \ t \geq 0\}$$

- $\mathbf{E}(A)$ = span of evaluation functionals $\widehat{x} \in \mathbf{V}(A)^*$, $x \in X(A)$, with cone

$$\mathbf{E}(A)_+ := \left\{ \sum_{i=1}^{k} t_i \widehat{x}_i \middle| x_i \in X(A), \ t_i \geq 0 \right\}$$

We assume $\dim \mathbf{V}(A) < \infty$. Then $\dim \mathbf{E}(A) = \dim \mathbf{V}(A)$.
Note that $\forall E \in \mathcal{M}(A)$,

$$\sum_{x \in E} \widehat{x} = u_A \in \mathbf{E}(A) \ \text{ where } \ u_A(\alpha) = 1 \ \forall \alpha \in \Omega(A)$$

Examples

Classical models: E a finite set: $A(E) = (E, \{E\}, \Delta(E))$, where $\Delta(E) =$ simplex of prob. weights on E. Then

$$\mathbf{V}(A) \simeq \mathbb{R}^E \simeq \mathbf{E}(A).$$

Quantum models: $\mathcal{H}$ a f.d. Hilbert space: Let $A(\mathcal{H}) = (X(\mathcal{H}), \mathcal{M}(\mathcal{H}), \Omega(\mathcal{H}))$ where $X(\mathcal{H}) =$ unit sphere, $\mathcal{M}(\mathcal{H}) =$ all orthonormal bases of $\mathcal{H}$, $\Omega(\mathcal{H})$ all prob. weights induced by density operators on $\mathcal{H}$. Then

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Goal (updated again): Conditions guaranteeing

- $\mathbf{E}(A)$ *self-dual,*
- $\mathbf{E}(A)_+ \simeq \mathbf{V}(A)_+$, *and*
- $\mathbf{V}(A)$ *homogeneous.*

Composite systems

A non-signaling composite of models A and B: a model AB, plus a mapping

$$X(A) \times X(B) \to \mathbf{E}_+(AB) : \quad (x, y) \mapsto xy$$

such that

(a) $(E, F) \in \mathcal{M}(A) \times \mathcal{M}(B) \implies \sum_{x \in E, y \in F} xy = u_{AB}$
(b) $\omega \in \Omega(AB) \implies \omega(x\ \cdot\ ) \in \mathbf{V}_+(B)$ and $\omega(\ \cdot\ y) \in \mathbf{V}_+(A)$

By (a), $\omega \in \Omega(AB)$ pulls back to a joint probability weight:

$$\omega(x, y) := \omega(xy); \quad \sum_{x \in E, y \in F} \omega(x, y) = 1.$$

AB is locally tomographic iff every state is uniquely determined by its corresponding joint probability weight.

By (b), $\omega \in \Omega(AB)$ has well-defined marginal and conditional states:

$$\omega_1(x) := \sum_{y \in F} \omega(\cdot, y) \quad \text{and} \quad \omega_{2|x}(y) := \frac{\omega(x,y)}{\omega_1(x)}.$$

both in $\Omega(A)$, and similarly for $\omega_2(y), \omega_{1|y} \in \Omega(B)$. This gives a Law of total probability: $\forall\ E \in \mathcal{M}(A)$, $F \in \mathcal{M}(B)$,

$$\omega_2 = \sum_{x \in E} \omega_1(x)\omega_{2|x} \quad \text{and} \quad \omega_1 = \sum_{y \in F} \omega_2(y)\omega_{1|y}$$

Lemma: $\omega \in \Omega(AB) \Rightarrow \exists!$ *positive linear mapping* $\widehat{\omega} : \mathbf{E}(A) \to \mathbf{V}(B)$, $\widehat{\omega}(x)(y) = \omega(x,y)\ \forall\ x \in X(A),\ y \in X(B)$.

Processes and categories

A process from A to B: is a positive linear mapping $\tau : \mathbf{V}(A) \to \mathbf{V}(B)$ such that

$$u_B(\tau(\alpha)) \leq 1$$

$\forall \alpha \in \Omega(A)$. τ is reversible iff invertible with ϕ^{-1} positive (so for some scalar $t > 0$, $t\phi^{-1}$ is a process.)

A monoidal probabilistic theory: a category $\mathcal{C}$ of probabilistic models and processes, symmetric-monoidal w.r.t. an operation $A, B \mapsto AB$ forming non-signaling composites.

From now, on, we work in a fixed monoidal theory $\mathcal{C}$.

(Mainly for convenience).

III. Conjugates and Filters

Conjugate quantum systems

Let $\mathcal{H}$ be a complex Hilbert space of dimension n. Consider the "EPR" state on $\mathcal{H} \otimes \overline{\mathcal{H}}$:

$$\Psi := \frac{1}{\sqrt{n}} \sum_{x \in E} x \otimes \overline{x},$$

where E is *any* orthonormal basis for $\mathcal{H}$ (ψ's independent of the choice!) For $a, b \in \mathbf{E}(\mathcal{H})$, one has

$$\mathrm{Tr}(ab) = \langle (a \otimes \overline{b})\psi, \psi \rangle.$$

So ψ *encodes the trace inner product on* $\mathbf{E}(\mathcal{H})$ *as a STATE* on $\mathcal{H} \otimes \overline{\mathcal{H}}$ — which *perfectly* and *uniformly* correlates *every* test $E \in \mathcal{M}(A)$ with its counterpart $\overline{\mathbf{E}} \in \mathcal{M}(\mathcal{H})$: $\forall x \in X(\mathcal{H})$,

$$|\langle \Psi, x \otimes \overline{x} \rangle|^2 = \frac{1}{n}.$$

Conjugates abstractly

Suppose all tests in $\mathcal{M}(A)$ have n outcomes.

A conjugate for A is a triple $(\overline{A}, \gamma_A, \eta_A)$ where $\gamma_A : A \simeq \overline{A}$ is an isomorphism and η_A is a non-signaling state on $A \otimes \overline{A}$ such that

(a) $\eta(x, \gamma_A(y)) = \eta(y, \gamma_A(x))$ and

(b) $\eta_A(x, \gamma_A(x)) = 1/n$ for every $x, y \in X(A)$.

Note that $(\eta_A)_1 = \rho$, the maximally mixed state $\rho(x) \equiv 1/n$.

Notation: From now on, $\gamma_A(x) = \overline{x}$.

Filters

Let A be a probabilistic model. A filter for a test E is a process $\phi : \mathbf{V}(A) \to \mathbf{V}(A)$ such that, for every $x \in E$ and every state $\alpha \in \Omega(A)$,

$$\phi(\alpha)(x) = t_x \alpha(x)$$

for some constant $0 \leq t_x \leq 1$. Call ϕ *symmetric* iff

$$(\phi \otimes \mathrm{id}_{\overline{A}})(\eta_A) = (\mathrm{id}_A \otimes \overline{\phi})(\eta_A).$$

Example: Let E be an orthonormal basis for $\mathcal{H}$: for any choice of $0 \leq t_x \leq 1$, $x \in E$, let $V = \sum_{x \in E} t_x P_x$. Set

$$\phi(a) = V^{1/2} a V^{1/2}$$

for $a \in \mathbf{E}(\mathcal{H})$. This is a filter, symmetric with respect to η described above. Since $\phi(\mathbf{1}) = V$, *any density operator V* can be prepared from the maximally mixed state, up to normalization, by such a filter — reversibly, if V is non-singular.

Conjugates and Filters

Theorem: *Suppose that, for every* $A \in \mathcal{C}$,

- *A has a conjugate,* $\overline{A} \in \mathcal{C}$.
- *Any non-singular state* $\alpha \in \Omega(A)$ *can be prepared from* ρ *(up to normalization) by a (reversible) symmetric filter.*

Then, for every $A \in \mathcal{C}$, $\mathbf{E}(A)$ *is HSD with respect to the inner product* $\langle a, b \rangle = \eta_A(a, \overline{b})$.

Proof: Conspicuously easy, using the KV theorem! See arXiv: 1206.2897.

Sketch of Proof (arXiv:1206:2897):

Homogeneity is clear.
Let $\alpha = \phi(\rho)$, ϕ a symmetric filter for $E \in \mathcal{M}(A)$. Set $\omega = (\phi \otimes \mathrm{id}_{\overline{A}})(\eta)$.
Then $\omega_1 = \phi(\rho) = \alpha$. For $x \in E$, set $\delta_x := \frac{1}{n}\eta(\cdot, \overline{x})$. As ϕ is symmetric,

$$\omega_{1|\overline{x}}(x)\omega_2(\overline{x}) = \eta(\cdot, \phi(\overline{x})) = t_x \eta(\cdot, \overline{x}) = t_x n \delta_x$$

The LOTP now gives us a kind of "spectral decomposition":

$$\alpha = \sum_{\overline{x} \in \overline{E}} t_x \delta_x \tag{1}$$

Thus, $\mathbf{V}(A)_+$ is generated by δ_x-s. So $\widehat{\eta} : \mathbf{E}(A) \to \mathbf{V}(A)$, given by $\widehat{\eta}(a)(x) = \eta(x, \overline{a})$, takes $\mathbf{E}(A)_+$ *onto* $\mathbf{V}(A)_+$. By dimensionality, $\widehat{\eta}$ is injective, so $\widehat{\eta} : \mathbf{E}(A) \simeq \mathbf{V}(A)$ (as ordered spaces).

Use this to pull the decomposition (1) back to one for elements of $\mathbf{E}(A)$.
Use *this* to show $\eta(a, \overline{b})$ is positive-semidefinite.

IV. Categories of Jordan models

The Hanche-Olsen tensor product

One can show that the only *locally tomographic* monoidal probabilistic theory satisfying these conditions is standard finite-dimensional $\mathbb{C}$-QM (Barnum, AW 2012)

Are there any other, non-LT probabilistic theories of this kind?

Let $\mathbf{E}$ be any euclidean Jordan algebra. There's a universal C^*-algebra $C^*(\mathbf{E})$ (jordan homomorphisms $\phi : \mathbf{E} \to \mathcal{A}_h$, $\mathcal{A}$ a C^* algebra, factor uniquely through $*$-homomorphisms $C^*(\mathbf{E}) \to \mathcal{A}$).

(*Note:* $\mathbf{E}$ exceptional implies $C^*(\mathbf{E}) = \mathbf{0}$.)

Definition [Hanche-Olsen]: If $\mathbf{E}$ and $\mathbf{F}$ are EJAs, let $\mathbf{E}\tilde{\otimes}\mathbf{F}$ = Jordan subalgebra of $[C^*(\mathbf{E}) \otimes C^*(\mathbf{F})]_h$ generated by $\mathbf{E} \otimes \mathbf{F}$.

A Sample Theory

Let $\mathcal{C}$ be the category having objects = euclidean Jordan models, morphisms $\mathbf{E} \to \mathbf{F}$ = restrictions of CP maps $C^*(\mathbf{E}) \to C^*(\mathbf{F})$, and composites given by $A(\mathbf{E})A(\mathbf{F}) = A(\mathbf{E}\tilde{\otimes}\mathbf{F})$. (This is a non-signaling composite.)

One can show that all Jordan models $A(\mathbf{E})$ satisfies hypothesis (b) of the Theorem. What about conjugates? Define

$$\gamma : C^*(\mathbf{E}) \to C^*(\mathbf{E})^{\mathrm{op}} \ \text{ and } \ a \mapsto (a^{op})^* := \overline{a}.$$

Setting $\overline{\mathbf{E}} = \gamma(\mathbf{E}) \leq C^*(\mathbf{E})^{\mathrm{op}}$, we have $C^*(\overline{\mathbf{E}}) = C^*(\mathbf{E})^{\mathrm{op}}$. The mapping $a, \overline{b} \mapsto \mathrm{Tr}(ab^*)$ defines a state

$$\widehat{\eta} : C^*(\mathbf{E}) \otimes C^*(\overline{\mathbf{E}}) \to \mathbb{C}$$

which restricts to a non-signaling state on $\mathbf{E}\tilde{\otimes}\overline{\mathbf{E}}$ making $(A(\overline{\mathbf{E}}), \gamma_A, \eta_A)$ a conjugate for $A(\mathbf{E})$.

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