Fuzzy Structures in Quantum Computation with Mixed States

Giuseppe Sergioli
Hector Freytes
Cagliari Logic Group
[giuseppe.sergioli@gmail.com]

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Continuous Triangular norm: Definition

A continuous trinagular norm (continuous t- norm, shortly) is a function $\star : [0,1]^2 \rightarrow [0,1]$ that safisfies the following properties:

- Commutativity: $x \star y = y \star x$
- Associativity: $(x \star y) \star z = x \star (y \star z)$
- Monotony: if $x \leq y$ then $x \star z \leq y \star z$
- Continuity

Fuzzy Logics associated to continuous t-norm

Each of the following t-norms is a natural representation of the *Conjunction* in the respective Logic

$$x \odot_P y = x \cdot y \qquad (\textit{Product } t\textit{-norm})$$
$$x \odot y = \max\{x + y - 1, 0\} \qquad (\textit{Łukasiewicz } t\textit{-norm})$$
$$x \odot_G y = \min\{x, y\} \qquad (\textit{Gödel } t\textit{-norm})$$

We will represent (in a probabilistic way) these t-norms in the framework of Quantum Computation with mixed states.

Łukasiewicz and Product *t*-norms are known for their relations with game theory applied to the theory of communication with feedback.

- Łukasiewicz *t*-norm is related to Ulam's games
- Product *t*-norm is specially applied in fuzzy control and allows us to model a probabilistic variant of Ulam' s game, the so called Pelc's game

Part I

Representing Product *t*-norm

Standard Quantum Computation

Standard Quantum Computation is based on:

- *qubit* i.e. a pure state in $\mathbb{C}^2$
- Unitary operators (quantum gates)

In general, a quantum system is not in a pure state (decoherence, environments, etc...). For this we need a powerfull model, where:

Quantum Computation with Mixed States

- Qubits are replaced by density operators
- Unitary operators are replaced by Quantum Operations

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger,$$

where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation)

Recalling the Born rule, one can naturally define the probability-value of any density operator ρ of $\mathcal{H}^{(n)} = \otimes^n \mathbb{C}^2$.

Probability

For any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$,

$$\mathrm{p}(\rho) := Tr(P_1^{(n)} \rho),$$

where $\mathfrak{D}(\mathcal{H}^{(n)})$ is the set of all density operators of $\mathcal{H}^{(n)}$.

Logical overview

Conventionally, we can assume that the two elements $|1\rangle$ and $|0\rangle$ of the canonical orthonormal basis of the Hilbert space $\mathbb{C}^2$ represent the canonical truth-values *Truth* and *Falsity*.

The notions of *truth*, *falsity* and *probability*:

True and false registers

- $|x_1, \ldots, x_n\rangle$ is a *true register* iff $|x_n\rangle = |1\rangle$;
- $|x_1, \ldots, x_n\rangle$ is a *false register* iff $|x_n\rangle = |0\rangle$.

In other words, the *truth-value* of a register is determined by its last element.

Truth and falsity

- The *truth property* of $\mathcal{H}^{(n)}$ is the projection operator $P_1^{(n)}$ that projects over the closed subspace spanned by the set of all true registers.
- The *falsity property* of $\mathcal{H}^{(n)}$ is the projection operator $P_0^{(n)}$ that projects over the closed subspace spanned by the set of all false registers.

The Toffoli Gate

For any $m, k, p \geq 1$, the Toffoli gate $T^{(m,k,p)}$ is defined on $\mathcal{H}^{(m+k+p)} = \otimes^m \mathbb{C}^2 \otimes \otimes^k \mathbb{C}^2 \otimes \otimes^p \mathbb{C}^2 = \otimes^{(m,k,p)} \mathbb{C}^2$ as follows.

If $|x\rangle = |x_1 \ldots x_m\rangle \in \otimes^m \mathbb{C}^2$, $|y\rangle = |y_1 \ldots y_k\rangle \in \otimes^k \mathbb{C}^2$ and $|z\rangle = |z_1 \ldots y_p\rangle \in \otimes^p \mathbb{C}^2$, then :

$$T^{(m,k,p)}(|x\rangle \otimes |y\rangle) \otimes |z\rangle) = |x\rangle \otimes |y\rangle \otimes |z_1, \ldots, z_{p-1}\rangle \otimes |x_m y_k \hat{\oplus} z_p\rangle$$

where $\hat{\oplus}$ is the sum modulo 2.

Matrix representation of Toffoli $T^{(m,k,1)}$

Let us consider a Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ where $dim(\mathcal{H}_a) = 2^m$ and $dim(\mathcal{H}_b) = 2^k$.
Then, the Toffoli gate $T^{(m,k,1)}$ can be seen as a *block diagonal* matrix:

$$T^{(m,k,1)} = I^{(2^{m-1} \times 2^{m-1})} \otimes \left[\begin{array}{c|c} I^{(2^{k+1} \times 2^{k+1})} & \mathbf{0} \\ \hline \mathbf{0} & I^{(2^{k-1} \times 2^{k-1})} \otimes Xor \end{array} \right]$$

where $Xor = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$

$$T^{(1,1,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Compositional Conjunction

Let ρ be a density operator of $\otimes^m \mathbb{C}^2$ and let σ be a density operator in $\otimes^k \mathbb{C}^2$. The *Compositional Conjunction* of ρ and σ is defined as follows:

$$AND^{(m,k)}(\rho \otimes \sigma) = {}^{\mathcal{D}}T^{(m,k,1)}(\rho \otimes \sigma \otimes P_0^{(1)})$$

where ${}^{\mathcal{D}}T^{(m,k,1)}$ is the "left-right" application of the Toffoli matrix. One can prove:

$$\mathrm{p}(AND^{(m,k)}(\rho \otimes \sigma)) = \mathrm{p}(\rho)\mathrm{p}(\sigma).$$

***AND* probabilistically represents the Product *t*-norm.**

But in real cases the input of *AND* can be an *unfactorized* state!

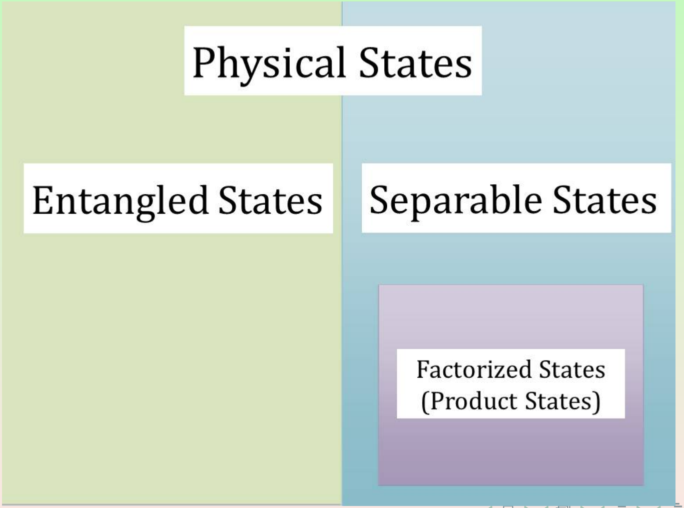
This requires a new conjunction, called:

Holistic Conjunction

Entanglement

The meaning of entanglement is strictly related to the principle of quantum non-separability.

Consider a density operator ρ of $\mathcal{H}^a \otimes \mathcal{H}^b$.
We say that ρ represents an *entangled state* iff ρ cannot be decomposed as a convex combination of density operators having the form: $\rho^a \otimes \rho^b$, with ρ^a density operator in $\mathcal{H}^a$ and ρ^b density operator in $\mathcal{H}^b$.
In other words:
$\rho \neq \sum_i \lambda_i \rho_i$, where λ_i are positive real numbers such that $\sum_i \lambda_i = 1$ and ρ_i are density operators having the form: $\rho^a \otimes \rho^b$.


Physical States
Entangled States
Separable States
Factorized States
(Product States)

Remark

Factorized states are only special cases of non-entangled states.

Example

Consider the state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Let $P_{|\psi^-\rangle}$ be the projection operator determined by $|\psi^-\rangle$ and let $I^{(2)}$ be the identity operator of $\mathcal{H}^{(2)}$.

Let $\rho = \frac{1}{3}P_{|\psi^-\rangle} + \frac{1}{6}I^{(2)}$.

Example

$$\rho = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

Even if ρ is not factorizable, ρ represents a non-entangled state!

Representation via Pauli Matrices

Let $\sigma_i, \{i = 1, 2, 3\}$ represent the Pauli matrices and let ρ be a density operator in the Hilbert space $\mathbb{C}^2$. Then:

$$\rho = \frac{1}{2}(I^2 + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)$$

- $|r_1|^2 + |r_2|^2 + |r_3|^2 = 1 \rightarrow$ Pure state
- $|r_1|^2 + |r_2|^2 + |r_3|^2 < 1 \rightarrow$ Mixed state

Let $\mathcal{H}$ be an n-dimensional Hilbert space an let $\{|\psi_j\rangle\}_{j=1}^{n}$ be an orthonormal basis of $\mathcal{H}$. Let us consider the following three families $\mathbf{B}, \mathbf{C}, \mathbf{D}$ of $n \times n$ matrices:

$$\mathbf{B} = \{B_k : 1 \leq k \leq n-1\}$$

where

$$B_k := \sqrt{\frac{2}{k(k+1)}}(|\psi_1\rangle\langle\psi_1| + \cdots + |\psi_k\rangle\langle\psi_k| - k|\psi_{k+1}\rangle\langle\psi_{k+1}|)$$

$$\mathbf{C} = \{C_{k,j} : 1 \leq k < j \leq n\}$$

where

$$C_{k,j} := |\psi_j\rangle\langle\psi_k| + |\psi_k\rangle\langle\psi_j|$$

$$\mathbf{D} = \{D_{k,j} : 1 \leq k < j \leq n\}$$

where

$$D_{k,j} = i(|\psi_j\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_j|)$$

Consider the set $\Sigma = \mathbf{C} \cup \mathbf{D} \cup \mathbf{B}$ ordered as follows

$$\begin{aligned}\Sigma &= \{C_{1,2}, C_{1,3}, \ldots, C_{2,3}, \ldots | D_{1,2}, D_{1,3}, \ldots, D_{2,3}, \ldots | B_1, \ldots, B_{n-1}\} \\ &= \{\sigma_1, \ldots, \sigma_{\frac{n(n-1)}{2}} | \sigma_{\frac{n(n-1)}{2}+1}, \ldots, \sigma_{n(n-1)} | \sigma_{n(n-1)+1}, \ldots, \sigma_{n^2-1}\}\end{aligned}$$

The elements of the sequence Σ are called *generalized Pauli Matrices*. The elements of Σ are the generators of $SU(n)$. In particular $Tr\Sigma_i = 0$ for any i and $Tr(\Sigma_i\Sigma_j) = 2\delta_j$.

Theorem

Let ρ be a density operator of an n-dimensional Hilbert space $\mathcal{H}$. Then:

$$\rho = \frac{1}{n} I^n + \frac{1}{2} \sum_{j=1}^{n^2-1} s_j(\rho)\sigma_j$$

where:

- σ_j *are the generalized Pauli matrices of* $\mathcal{H}$*;*
- $s_j(\rho) = tr(\rho\sigma_j)$. *The sequence* $\langle s_1(\rho) \ldots s_{n^2-1}(\rho)\rangle$ *is called the generalized Bloch vector associated to* ρ.

Example

For $n = 2$ we obtain the usual representation of a density operator in $\mathbb{C}^2$:

$$\rho = \frac{1}{2}(I^2 + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3).$$

Matrix representation of the partial trace

Let ρ be a density operator of an n-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ where $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = k$. If we divide ρ in $m \times m$ blocks $B_{i,j}$, where each block is a $k \times k$ matrix, then:

$$\begin{aligned}
\rho^a &= tr_b\rho = \begin{pmatrix} trB_{1,1} & trB_{1,2} & \dots & trB_{1,m} \\ trB_{2,1} & trB_{2,2} & \dots & trB_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ trB_{m,1} & trB_{m,2} & \dots & trB_{m,m} \end{pmatrix} \\
\rho^b &= tr_a\rho = \sum_{i=1}^{m} B_{i,i}
\end{aligned}$$

Theorem

Let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ where $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = n$. Consider $\sigma_1^a \ldots \sigma_{m^2-1}^a$ and $\sigma_1^b \ldots \sigma_{n^2-1}^b$, the generalized Pauli matrices of $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. Then, any density operator ρ of $\mathcal{H}$ can be represented as follows:

$$\rho = \rho^a \otimes \rho^b + Fac(\rho).$$

where $Fac(\rho) = \frac{1}{4} \sum_{j=1}^{m^2-1} \sum_{k=1}^{n^2-1} fac_{j,k}(\rho)(\sigma_j^a \otimes \sigma_k^b)$ *and*

$$fac_{j,k}(\rho) = tr(\rho[\sigma_j^a \otimes \sigma_k^b]) - tr(\rho[\sigma_j^a \otimes I^n])tr(\rho[I^m \otimes \sigma_k^b])$$

Hence, any ρ can be represented as a sum of a factorized state and a paricular self-adjoint operator.
(The Schlienz-Mahler decomposition).

One can prove that:

$$Tr(P_1 Fac(\rho)) = 0$$

The Holistic Conjunction

Let ρ be a density operator of $\mathcal{H} = \otimes^m \mathbb{C}^2 \otimes \otimes^k \mathbb{C}^2 = \otimes^{(m,k)} \mathbb{C}^2$.

The *Holistic Conjunction* $AND_{Hol}^{(m,k)}$ on ρ is defined as follows:

$$AND_{Hol}^{(m,k)}(\rho) = {}^{\mathcal{D}}T^{(m,k,1)}(\rho \otimes P_0^{(1)})$$

The probability of the *Holistic Conjunction*

Theorem

Let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, *where* $\mathcal{H}_a = \otimes^m \mathbb{C}^2$ *and* $\mathcal{H}_b = \otimes^k \mathbb{C}^2$. *Then,*

$$\mathrm{p}(AND^{(m,k)}_{Hol}(\rho)) = \mathrm{p}(\rho^a)\mathrm{p}(\rho^b) + fac(\rho, AND^{(m,k)}_{Hol}) =$$

$$= \mathrm{p}(AND(\rho^a \otimes \rho^b)) + fac(\rho, AND^{(m,k)}_{Hol})$$

where
$fac(\rho, AND^{(m,k)}_{Hol}) =$
$\frac{1}{4} \sum_{j=2^m(2^m-1)+1}^{2^{2m}-1} \sum_{i=2^k(2^k-1)+1}^{2^{2k}-1} fac_{j,i}(\rho) tr(P_1^{(m+k)}(\sigma_j^a \otimes \sigma_i^b))$

We have:

- $-\frac{1}{4} \leq fac(\rho, AND_{Hol}^{(m,k)}) \leq \frac{1}{4}$;
- If ρ is factorizable then $fac(\rho, AND_{Hol}^{(m,k)}) = 0$ (but the other way around does not hold).

The Holisitc Conjunction does not caracterize the Entanglement!

The probability of a Holistic Conjunction

Let ρ be a density operator of $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b = \otimes^{(m,k)}\mathbb{C}^2$.
Let us indicate by a_i the i-th diagonal element of ρ.
We have:

$$\mathrm{p}(AND_{Hol}^{(m,k)}(\rho)) = \sum_{\alpha=1}^{2^{m-1}} \sum_{\beta=1}^{2^{k-1}} a_{(2\alpha-1)2^k+2^\beta}$$

Example

Let $\rho \in \mathcal{H}^a \otimes \mathcal{H}^b$ with $\mathcal{H}^a = \mathcal{H}^b = \mathbb{C}^2$ and let us indicate with a_i the i-th diagonal element of ρ. Then,

$$\mathrm{p}(AND_{Hol}^{(1,1)}(\rho)) = \sum_{\alpha=1}^{1} \sum_{\beta=1}^{1} a_{(2\alpha-1)2^1+2^\beta} = a_4$$

The Holistic Conjunction

Remark

Suppose that $m \neq m'$, $k \neq k'$, $m + k = m' + k' = n$ and let ρ be a density operator of $\otimes^n \mathbb{C}^2$.

Generally we have:

$$\mathtt{p}(AND_{Hol}^{(m,k)}(\rho)) \neq \mathtt{p}(AND_{Hol}^{(m',k')}(\rho)).$$

Matrix block representation of the Holistic Conjunction

Let ρ be a density matrix of $\mathcal{H}^a \otimes \mathcal{H}^b = \otimes^{(m,k)}\mathbb{C}^2$.
On the diagonal of ρ we can individuate 2^m blocks consisting of $(2^k \times 2^k)$-matrices:

$$\rho = \begin{pmatrix} [\cdot \cdot]_1^{(2^k \times 2^k)} & & & \\ & [\cdot \cdot]_2^{(2^k \times 2^k)} & & \\ & & \ddots & \\ & & & [\cdot \cdot]_{2^m}^{(2^k \times 2^k)} \end{pmatrix}$$

Matrix blocks representation of the Holistic Conjunction

Let:

- α be the sum of the even diagonal elements of the even diagonal blocks of ρ;
- β be the sum of the odd diagonal elements of the even diagonal blocks of ρ;
- γ be the sum of the even diagonal elements of the odd diagonal blocks of ρ;
- δ be the sum of the odd diagonal elements of the odd diagonal blocks of ρ.

Matrix blocks representation of the Holistic Conjunction

We have the following results:

- $\mathrm{p}(\rho^{a}) = \alpha + \beta$;
- $\mathrm{p}(\rho^{b}) = \alpha + \gamma$;
- $\mathrm{p}(AND_{Hol}^{(m,k)}(\rho)) = \alpha$.

Hence,

$$fac(\rho, AND_{Hol}^{(m,k)}) = \alpha\delta - \beta\gamma.$$

Example

Consider the following density matrix ρ of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\rho = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

We obtain:

$$fac(\rho, AND^{(1,1)}_{Hol}) = -\frac{1}{12}.$$

The Holistic Conjunction

Theorem

Let ρ be a density operator of $\mathcal{H} = \mathcal{H}^a \otimes \mathcal{H}^b = \otimes^{(m,k)}\mathbb{C}^2$.

1. $\mathrm{p}(AND_{Hol}^{(m,k)}(\rho)) \leq \mathrm{p}(\rho^a), \mathrm{p}(\rho^b)$;
2. if $\mathrm{p}(AND_{Hol}^{(m,k)}(\rho)) = 1$, then $\mathrm{p}(\rho^a) = \mathrm{p}(\rho^b) = 1$ and consequently $fac(\rho, AND_{Hol}^{(m,k)}) = 0$;
3. the following situation is possible: $\mathrm{p}(\rho^a) \neq 0$, $\mathrm{p}(\rho^b) \neq 0$ and $\mathrm{p}(AND_{Hol}^{(m,k)}(\rho)) = 0$.

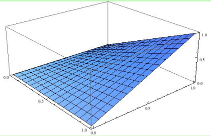


Figure : Compositional Conjunction

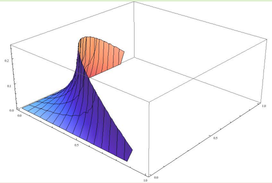


Figure : Holistic Conjunction

Werner states

Let $\rho_W^{[n]}$ be a density matrix of a Hilbert space $\mathcal{H}$ whose dimension is n^2 (with $n > 1$).
$\rho_W^{[n]}$ is called a Werner State of $\mathcal{H}$ iff, for any unitary operator U^n:

$$\rho_W^{[n]} = (U^n \otimes U^n)\rho_W^{[n]}((U^n)^\dagger \otimes (U^n)^\dagger).$$

Parametrization of a Werner State

Werner states can be parametrized in different ways.
One way is the following:

$$\rho^{[n]}_{W_{(\alpha)}} = \frac{n+1-2\alpha}{n(n^2-1)} I^{n^2} - \frac{n+1-2\alpha n}{n(n^2-1)} Sw^{n^2}$$

where I^{n^2} is the $n^2 \times n^2$ identity matrix and Sw^{n^2} is the $n^2 \times n^2$ Switch gate, given by: $Sw^{n^2} = \sum_{i,j}(|i\rangle\langle j| \otimes |j\rangle\langle i|)$, where $|i\rangle$ and $|j\rangle$ are vectors of the n-dimensional computational basis and α is a real number such that $\alpha \in [0, 1]$.

Theorem

Let $\rho^{[n]}_{W_{(\alpha)}}$ be a n^2-dimensional Werner state. Then:

i) $\rho^{[n]}_{W_{(\alpha)}}$ is **factorizable** iff $\alpha = \frac{n+1}{2n}$.

ii) $\rho^{[n]}_{W_{(\alpha)}}$ is **separable** iff $\frac{1}{2} \leq \alpha \leq 1$;

The real number α can be considered as related to a measure of entanglement (accordingly, the “degree of entanglement" of a Werner State is **inversely proportional** to α).

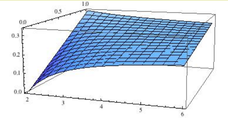
When $\alpha = 0$, we can reasonably assume that $\rho^{[n]}_{W_{(\alpha)}}$ is **maximally entangled**.

Theorem

Consider a Wernes State $\rho^{[n]}_{W_{(\alpha)}}$.

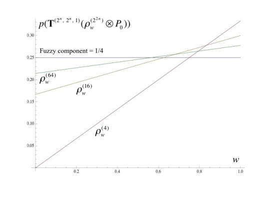
By using the matrix representation of $\rho^{[n]}_{W_{(\alpha)}}$, we obtain:

$\mathrm{p}(AND^{(n,n)}_{Hol}(\rho^{[n]}_{W_{(\alpha)}})) = \frac{n^2+n(2\alpha-1)-2}{4(n^2-1)}$;



Theorem

$$fac(\rho^{[n]}_{W_{(\alpha)}}, AND^{(n,n)}_{Hol}) = \frac{2\alpha n - n - 1}{4(n^2-1)}.$$



$fac(\rho^{[n]}_{W_{(\alpha)}}, AND^{(n,n)}_{Hol}) = 0$ iff the Werner state is non-factorizable .

The Holistic conjunction caracterizes Entanglement for Werner (and Isotropic) states!

Part II

Representing Łukasiewicz t-norm

$$x \odot y = \max\{x + y - 1, 0\}$$

k-order Polynomial - Notation

- The term *multi-index* denotes an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non negative integers α_i
- If k is a natural number, $\alpha \leq k$ means that $\alpha_i \leq k$ for each $i \in \{1, \ldots, n\}$
- The *order* of α is given by $|\alpha| = \alpha_1 + \ldots + \alpha_n$
- If $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index, the monomial $\mathbf{x}^\alpha$ is defined by $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

In this language a polynomial of order k is a function

$$P(\mathbf{x}) = \sum_{|\alpha| \leq k} a_\alpha \mathbf{x}^\alpha \quad s.t. \;\; a_\alpha \in \mathbb{R}$$

n-degree Bernstein polynomial basis

Let $\mathbf{x} = (x_1, \ldots, x_n)$, k be a natural number and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index such that $\alpha \leq k$. Then the Bernstein polynomial $B_{k,\alpha}(\mathbf{x})$ is defined as:

$$B_{k,\alpha}(\mathbf{x}) = \prod_{i=1}^{n} \binom{k}{\alpha_i} (1 - x_i)^{k-\alpha_i} x_i^{\alpha_i}.$$

Theorem

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and k be a positive integer. For any continuous function $f : [0,1]^n \to \mathbb{R}$ the polynomials

$$B_k(f, \mathbf{x}) = \sum_{\alpha \leq k} f\left(\frac{\alpha_1}{k}, \ldots, \frac{\alpha_n}{k}\right) B_{k,\alpha}(\mathbf{x})$$

converge to $f(\mathbf{x})$ uniformly on $[0,1]^n$ when $k \to \infty$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and k be a natural number. The Bernstein basis is given by:

$$\mathcal{B}_k(\mathbf{x}) = \{(1-x_1)^{\alpha_1} x_1^{\beta_1} \ldots (1-x_n)^{\alpha_n} x_n^{\beta_n} : \alpha_i + \beta_i = k, \ \ i \in \{1, \ldots, n\}\}$$

Bernstein basis and density operators

If $\rho = \begin{bmatrix} 1-x & \\ & x \end{bmatrix}$ and $\sigma = \begin{bmatrix} 1-y & \\ & y \end{bmatrix}$ are density operators, the diagonal of $\rho \otimes \sigma$ is the Bernstein basis $\mathcal{B}_2(x, y)$. In fact

$$\begin{bmatrix} 1-x & \\ & x \end{bmatrix} \otimes \begin{bmatrix} 1-y & \\ & y \end{bmatrix} = \begin{bmatrix} (1-x)(1-y) & & & \\ & (1-x)y & & \\ & & x(1-y) & \\ & & & xy \end{bmatrix}$$

Proposition

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a family of density operators such that

$$\mathbf{X}_i = \begin{pmatrix} 1 - x_i & b_i \\ b_i^* & x_i \end{pmatrix}$$

and let us consider a tensor product
$\mathbf{X} = (\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \ldots \otimes (\otimes^k \mathbf{X}_n)$. Then we have:

$$Diag(\mathbf{X}) = \mathcal{B}_k(x_1, \ldots, x_n)$$

where $Diag(\mathbf{X})$ denotes the set containing the diagonal entries of $\mathbf{X}$.

Polynomial quantum operation - Definition

A quantum operation $\mathcal{P} : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ is called *polynomial quantum operation* iff there exists a polynomial $P(x_1, \ldots, x_n)$ such that for each n-tuple $(\sigma_1, \ldots, \sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$ we have that:

$$p(\mathcal{P}((\otimes^k\sigma_1) \otimes \ldots \otimes (\otimes^k\sigma_n))) = P(p(\sigma_1), \ldots, \mathrm{p}(\sigma_n))$$

Theorem

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an n-tuple of variables and consider the set $\mathcal{B}_k(\mathbf{x})$.
Let $P(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{B}_k(\mathbf{x})} a_{\mathbf{y}} \mathbf{y}$ be a polynomial such that $\mathbf{y} \in \mathcal{B}_k(\mathbf{x})$, $0 \leq a_{\mathbf{y}}$ and $0 \leq P(\mathbf{x}) \mid_{[0,1]^n} \leq 1$. Then,

there exists a polynomial quantum operation $\mathcal{P} : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ such that for each n-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$

$$p(\mathcal{P}((\otimes^k \sigma_1) \otimes \ldots \otimes (\otimes^k \sigma_n))) = P(p(\sigma_1), \ldots, p(\sigma_n))$$

Stone Weierstrass type theorem

Let $f : [0,1]^n \to [0,1]$ be a continuous function. Then for each $\epsilon > 0$ there exists a quantum operation $\mathcal{P}_\epsilon : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ such that for each $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$,

$$|p(\mathcal{P}_\epsilon((\otimes^k \sigma_1) \otimes \ldots \otimes (\otimes^k \sigma_n))) - f(p(\sigma_1) \ldots p(\sigma_n))| \leq \epsilon$$

A problem

The convergence velocity by using Bernstein polynomial is low. It implies that: a good approximation need a high tensorial power. It is inefficient to implement in view of the fact that, it requires many copies of the involved states.

1. we introduce the auxiliary function

$$[0,2] \ni z \mapsto g(z) = \min(1,z) \quad s.t. \quad x \oplus y = g(x+y)$$

then the problem of approximating the bivariate function $\oplus$ is changed into the easier problem of approximating the one-variable function $g(z)$ in $[0,2]$.

2. By considering the function

$$[0,2] \ni z \mapsto h(z) = \begin{cases} \frac{z}{2}, & if \quad x \in [0,1] \\ 1 - \frac{z}{2}, & if \quad x \in (1,2] \end{cases}$$

3. We define

$$g(z) = \frac{z}{2} + h(z)$$

4. $h(z)$ is symmetric with respect to the point $z = 1$, i.e., $h(2-z) = h(z)$. For this reason we approximate $h(z)$ by using the symmetric functions

$$z^i(2-z)^i$$

5.

$$g_n(z) = \frac{z}{2} + \sum_{i=1}^{n} c_i z^i (2-z)^i, \qquad z \in [0,2]$$

6. The coefficients c_i are given by $\frac{-1^{i+1}}{2}\binom{1/2}{i}$

The approximant and the error

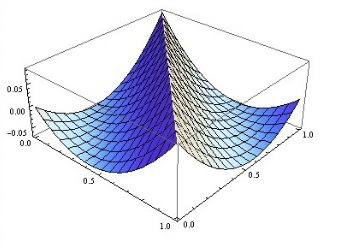
$$g_n(x+y) = \frac{x+y}{2} + \sum_{i=1}^{n} \frac{-1^{i+1}}{2} \binom{1/2}{i} (x+y)^i((1-x)+(1-y))^i$$

We estimate a bound for the approximation error, in fact:

$$e_n = \max_{x,y \in [0,1]} |(x \oplus y) - g_n(x+y)| \leq \frac{1}{2\sqrt{\pi n}} + O(n^{-3/2})$$

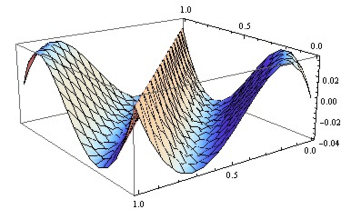
Case $n = 1$

- $g_1(x + y) = \frac{5}{12}(x + y)(1 - x) + \frac{5}{12}(x + y)(1 - y) + \frac{1}{2}(x + y)$
- error $\leq 0, 08$



Case $n = 2$

- $g_2(x+y) = \frac{1485}{2970}(x+y) + \frac{21}{2970}(x+y)((1-x)+(1-y)) + \frac{1372}{2970}(x+y)^2((1-x)^2 + 2(1-x)(1-y) + (1-y)^2)$
- error $\leq 0,04$



Part III

Quantum computational logic: probabilistic approach

Quantum computational logics with mixed states may be presented as a logic $\langle Term, \models \rangle$, where

- *Term* is an absolute free algebra (i.e. a language), whose natural universe of interpretation is a set $\mathcal{D}$ of density operators and whose connectives are naturally interpreted as certain quantum gates.
- *canonical interpretations* are *Term*-homomorphisms $e : Term \to \mathcal{D}$.
- *canonical valuations* are functions $f : Term \to [0, 1]$ such that f can be factorized in the following way:

$$\begin{array}{ccc} Term & \xrightarrow{\;f\;} & [0,1] \\ e \downarrow & \equiv & \nearrow p \\ \mathcal{D} & & \end{array}$$

  where p is the probability value $p(-) = tr(P_1 -)$.
- $\models$ is the logical consequence,

$$\alpha \models \beta \quad iff \quad p(\alpha) = 1 \Longrightarrow p(\beta) = 1$$

An *MV-algebra* is an algebra $\langle A, \oplus, \neg, 0 \rangle$ of type $\langle 2, 2, 0 \rangle$ satisfying the following equations:

- MV1 $\langle A, \oplus, 0 \rangle$ is an abelian monoid,
- MV2 $\neg\neg x = x$,
- MV3 $x \oplus \neg 0 = \neg 0$,
- MV4 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In agreement with the usual *MV*-algebraic operations we define:

$$x \odot y = \neg(\neg x \oplus \neg y), \qquad x \to y = \neg x \oplus y, \qquad x \vee y = (x \to y) \to y$$
$$x \wedge y = x \odot (x \to y), \qquad 1 = \neg 0,$$

A very important example of *MV*-algebra is

$$[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$$

such that $[0,1]$ is the real unit segment and $\oplus$ and $\neg$ are defined as follows:

$$x \oplus y = \min(1, x + y) \qquad \neg x = 1 - x$$

The derivate operations in $[0,1]_{MV}$ are given by

1. $x \odot y = \max(0, x + y - 1)$ (*Łukasiewicz t-norm*)
2. $x \to y = \min(1, 1 - x + y)$ (*Łukasiewicz implication*)

A *product MV-algebra* (for short: *PMV*-algebra) is an algebra $\langle A, \oplus, \bullet, \neg, 0 \rangle$ of type $\langle 2, 2, 1, 0 \rangle$ satisfying the following:

1. $\langle A, \oplus, \neg, 0 \rangle$ is an *MV*-algebra,
2. $\langle A, \bullet, 1 \rangle$ is an abelian monoid,
3. $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg (x \bullet z)$.

An important example of *PMV*-algebra is $[0, 1]_{MV}$ equipped with the usual multiplication i.e.

$$[0, 1]_{PMV} = \langle [0, 1], \oplus, \bullet, \neg, 0 \rangle$$

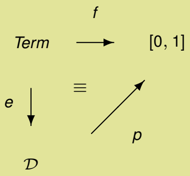
Proposition

Each *PMV*-algebra is isomorphic to a subdirect product of linearly ordered *PMV*-algebras.

1. Let *Term* be and absolutely free algebra in the signature $\langle \oplus, \cdot, \neg, 0 \rangle$
2. Let $\mathcal{D}$ be a set of density operators closed by $\langle \oplus, IAND, NOT \rangle$ such that $\frac{1}{tr(P_0)} P_0 \in \mathcal{D}$
3. Let $e : Term \to \mathcal{D}$ be a $\langle \oplus, \cdot, \neg, 0 \rangle$-Homomorphisms

Theorem

Let us consider the diagram of canonical valuations.



1. $\ker(p)$ *is a congruence respect to* $\langle \oplus, IAND, NOT, \frac{1}{tr(P_0)} P_0 \rangle$
2. $\mathcal{D}/_{Ker(p)}$ *is PMV-isomorphic to* $[0,1]_{PMV}$.

Question

What is the algebra associated to $\langle \mathcal{D}, \oplus, IAND, NOT, \frac{1}{tr(P_0)} P_0 \rangle$?

- This algebra describes the combinational logic of the quantum gates $\langle \oplus, IAND, NOT \rangle$.
- It plays a similar role that Boolean algebra describing the combinational logic for digital circuits.

The first and more basic algebraic structure associated to $\langle \oplus, NOT \rangle$ is the *quasi MV-algebra* or *qMV*-algebra for short. It is an algebra $\langle A, \oplus, \neg, 0, 1 \rangle$ of type $\langle 2, 1, 0, 0 \rangle$ satisfying the following equations:

Q1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

Q2. $\neg\neg x = x$,

Q3. $x \oplus 1 = 1$,

Q4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

Q5. $\neg(x \oplus 0) = \neg x \oplus 0$,

Q6. $(x \oplus y) \oplus 0 = x \oplus y$,

Q7. $\neg 0 = 1$.

From an intuitive point of view, a *qMV*-algebra can be seen as an *MV*-algebra which fails to satisfy the equation $x \oplus 0 = x$.

Let A be a quasi PMV-algebra

- $a \equiv_0 b$ iff $a \oplus 0 = b \oplus 0$ is a congruence in A
- $A/\equiv_0$ is a PMV-algebra
- If A is a set of density operators

$$\equiv_0 = Ker(p) \quad where \;\; p(-) = tr(P_1(-))$$

Note that

The natural projection $\pi : A \to A/\equiv_0$ is the abstract description of the probability $p(-) = tr(P_1(-))$

$\equiv_0$ is a uniformly defined congruence in the category $q\mathcal{PMV}$. Hence it defines a reflector

$$q\mathcal{PMV} \xrightarrow{\Pi} \mathcal{PMV} \quad where \;\; A \mapsto \Pi(A) = A/{\equiv_0}$$

Hence the Logic that describes the logical consequence

$$\alpha \models \beta \quad iff \quad p(\alpha) = 1 \Longrightarrow p(\beta) = 1$$

is the PMV-calculus

PMV-calculus

Łukasiewicz axioms:

W1 $\alpha \to (\beta \to \alpha)$,

W2 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,

W3 $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,

W4 $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$,

Product axioms:

P1 $(\alpha \bullet \beta) \to (\beta \bullet \alpha)$,

P2 $(1 \bullet \alpha) \leftrightarrow \alpha$,

P3 $(\alpha \bullet \beta) \to \beta$,

P4 $((\alpha \bullet \beta) \bullet \gamma) \leftrightarrow (\alpha \bullet (\beta \bullet \gamma))$,

P5 $(\alpha \bullet (\beta \odot \neg\gamma)) \leftrightarrow ((\alpha \bullet \beta) \odot \neg(\alpha \bullet \gamma))$,

The deduction rule is *modus ponens*

$$\{\alpha, \alpha \to \beta\} \vdash \beta$$

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Thank You!