Quantum Logic: Homework 3

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Objective

The primary objective of this homework is develop an understanding of the Logic of Quantum Programs and the Probabilistic Logic of Quantum Programs.

Preliminaries

⋆⋆ *Axiomatization of Orthologic and Orthomodular Quantum Logic*

Axiomatizations for orthologic and orthomodular quantum logic can be found in [4, 5, 6]. Here is a natural deduction style axiomatization of orthologic similar to the one given in [4, 5].

$$T \cup \{\varphi\} \vdash \varphi \qquad \frac{T \vdash \varphi, R \cup \{\varphi\} \vdash \psi}{T \cup R \vdash \psi}$$

$$T \cup \{\varphi \wedge \psi\} \vdash \varphi \qquad T \cup \{\varphi \wedge \psi\} \vdash \psi$$

$$\frac{T \vdash \varphi, T \vdash \psi}{T \vdash \varphi \wedge \psi} \qquad \frac{T \cup \{\varphi, \psi\} \vdash \chi}{T \cup \{\varphi \wedge \psi\} \vdash \chi}$$

$$T \cup \{\varphi\} \vdash \neg\neg\varphi \qquad T \cup \{\neg\neg\varphi\} \vdash \varphi$$

$$\frac{\{\varphi\} \vdash \psi, \{\varphi\} \vdash \neg\psi}{\emptyset \vdash \neg\varphi} \qquad T \cup \{\varphi \wedge \neg\varphi\} \vdash \psi \qquad \frac{\{\varphi\} \vdash \psi}{\{\neg\psi\} \vdash \neg\varphi}$$

An axiomatization of orthomodular quantum logic consists of all the rules of orthologic together with the following

$$\varphi \wedge (\neg\varphi \vee (\varphi \wedge \psi)) \vdash \psi$$

See [4, 5] for the meaning of $T \vdash \varphi$ and $T \models \varphi$.

⋆⋆ *Characterization of certain types of Hilbert lattices*

A *bounded lattice* is a lattice with a greatest element I ("top") and a least element O ("bottom"). An *ortholattice* $\mathbb{L}$ is a bounded lattice $(L, \leq)$ that satisfies (1) below. An *orthomodular lattice* $\mathbb{L}$ is an ortholattice $(L, \leq, -)$ that satisfies (2). A *propositional system* $\mathbb{L}$ is an orthomodular lattice $(L, \leq, -)$ that satisfies (3)–(5). Lastly, a *Piron lattice* $\mathbb{L}$ is a propositional system $(L, \leq, -)$ that satisfies (6).

1. **Orthocomplement:** $\mathbb{L}$ is equipped with a map $- : L \to L$ such that
   (a) $-(-p) = p$;
   (b) $p \leq q$ implies $-q \leq -p$;
   (c) $p \wedge -p = \mathbf{0}$ and $p \vee -p = \mathbf{1}$.

2. **Weak Modularity:** $q \le p$ implies $p \wedge (-p \vee q) = q$

3. **Completeness:** For any $A \subseteq L$, its meet $\bigwedge A$ and join $\bigvee A$ are in L.

Call $a \in L$ an *atom* if $a \neq \mathbf{0}$ and either $p = \mathbf{0}$ or $p = a$ holds for every $p \in L$ such that $p \le a$.

4. **Atomicity:** For any $p \neq \mathbf{0}$, there is an atom a such that $a \le p$.

5. **Covering Law:** If a is an atom and $a \not\le -p$ then $p \wedge (-p \vee a)$ is an atom.

6. **Superposition Principle:** For any two distinct atoms a, b, there is an atom c, distinct from both a and b, such that $a \vee c = b \vee c = a \vee b$.

A Piron lattice is usually defined with the superposition principle replaced by the condition that the lattice is irreducible: the lattice is not a direct sum of two sub lattice each with at least two elements. A direct sum between involuted lattice $\mathbb{L}_1 = (L_1, \le_1 -_1)$ and $\mathbb{L}_2 = (L_2, \le_2, -_2)$ is the involuted lattice $\mathbb{L} = (L, \le, -)$, where

1. $L = L_1 \times L_2$

2. $(a_1, b_1) \le (a_2, b_2)$ if and only if $a_i \le b_i$ for each $i \in \{1, 2\}$.

3. $-(a, b) = (-a, -b)$.

Mayet's condition states there is an automorphism T that maps closed linear subspaces onto closed linear subspaces, such that

1. There is a closed linear subspace Y, such that $T(Y) \subseteq Y$.

2. There is a closed linear subspace Y, such that there are at least two distinct subspaces Z_1 and Z_2, such that $Z_1 \subsetneq Y$ and $Z_2 \subsetneq Y$ and for all closed subspaces $Z \subseteq Y$, $T(Z) = Z$.

For any Piron lattice, there is a generalized Hilbert space (a generalization of a Hilbert space over a division ring — see [1, Section 8.1]), such that the lattice of bi-orthogonally closed subspaces of the generalized Hilbert space is isomorphic to the Piron lattice. A Piron lattice that also satisfies Mayet's condition is a Hilbert lattice for an infinite dimensional Hilbert space over the complex numbers, the real numbers, the division ring of the quaternions.

⋆⋆ *Logic for Quantum Programs*

Quantum dynamic frames

A *quantum dynamic frame* $\mathfrak{F}$ is a tuple $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ such that Σ is a set, $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$, and $\xrightarrow{P?} \subseteq \Sigma \times \Sigma$ for each $P \in \mathcal{L}$, and that satisfies the following, where $\to = \bigcup_{P \in \mathcal{L}} \xrightarrow{P?}$:

1. $\mathcal{L}$ is closed under arbitrary intersection.

2. $\mathcal{L}$ is closed under orthocomplement, where the orthocomplement of $A \subseteq \Sigma$ is $\sim A := \{s \in \Sigma \mid s \not\to t \text{ for all } t \in A\}$.

3. **Atomicity:** For any $s \in \Sigma$, $\{s\} \in \mathcal{L}$.

4. **Adequacy:** For any $s \in \Sigma$ and $P \in \mathcal{L}$, if $s \in P$, then $s \xrightarrow{P?} s$.

5. **Repeatability:** For any $s, t \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t$, then $t \in P$.

6. **Self-Adjointness:** For any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t \to u$, then there is a $v \in \Sigma$ such that $u \xrightarrow{P?} v \to s$.

7. **Covering Property:** Suppose $s \xrightarrow{P?} t$ for $s, t \in \Sigma$ and $P \in \mathcal{L}$. Then, for any $u \in P$, if $u \neq t$ then $u \to v \not\to s$ for some $v \in P$; or, contrapositively, $u = t$ if $u \to v$ implies $v \to s$ for all $v \in P$.

8. **Proper Superposition:** For any $s, t \in \Sigma$ there is a $u \in \Sigma$ such that $s \to u \to t$.

Given a Piron lattice $\mathfrak{L} = (L, \leq, -)$, let $F(\mathfrak{L}) = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$ be defined by

1. Σ is the set of atoms of $\mathfrak{L}$

2. $\mathcal{L}$ is the set $\{\{a \mid a \leq p, a \text{ is an atom}\} \mid p \in L\}$.

3. For each $x \in \mathcal{L}$, where $p = \bigvee x$, define $\xrightarrow{x?} \subseteq \Sigma \times \Sigma$ by $a \xrightarrow{x?} b$ if and only if $p \wedge (-p \vee a) = b$.

Then $F(\mathfrak{L})$ is a quantum dynamic frame (see [3]).

Given a quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$, let $G(\mathfrak{F}) = (\mathcal{L}, \subseteq, \sim)$, where $\sim A = \{s \mid s \not\to t, \forall t \in A\}$. Then $G(\mathfrak{F})$ is a Piron lattice (see [3]).

Language for quantum actions and semantics

Here is a language similar to the logic for quantum programs given in [2]. The language is two sorted, with formulas and programs, and is the same as propositional dynamic logic. Formulas are given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [\pi]\varphi$$

and the programs are given by

$$\pi := \varphi? \mid \pi_1 \cup \pi_2 \mid \pi_1; \pi_2$$

This language often appears with some constant states and constant actions. One type of action is the unitary, such as the Hadamard transform. These are building blocks for quantum programs. The semantics is often given on the closed linear subspaces of a Hilbert space, but for this section we will define the semantics directly on quantum Kripke models.

A quantum Kripke model is a tuple $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}}, V)$, where $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$ is a quantum Kripke frame and $V : \Phi \to \mathcal{L}$ is a valuation of atomic propositions to the set of testable properties $\mathcal{L}$. We interpret formulas in a quantum Kripke model by mapping formulas φ to subsets $[\![\varphi]\!]$ of Σ. These subsets that φ may denote need not all be in $\mathcal{L}$. However, when performing tests, we ensure what is being tested is in $\mathcal{L}$. For any subset S of Σ, let $\mathrm{cl}(S)$ be the smallest set in $\mathcal{L}$ containing S (such a smallest set exists as $\mathcal{L}$ is closed under intersection, and is equal to $\sim\sim S$). Also, for elements $S, T \in \mathcal{L}$, let $S \sqcup T = \mathrm{cl}(S \cup T)$. The programs are interpreted as relations $[\![\pi]\!] \subseteq \Sigma \times \Sigma$. These are done in the standard way as in propositional dynamic logic.

$[\![p]\!]$	$=$	$V(p)$
$[\![\neg\varphi]\!]$	$=$	$\Sigma \setminus [\![\varphi]\!]$
$[\![\varphi_1 \wedge \varphi_2]\!]$	$=$	$[\![\varphi_1]\!] \cap [\![\varphi_2]\!]$
$[\![[\pi]\varphi]\!]$	$=$	$\{s \mid t \in [\![\varphi]\!] \text{ whenever } s[\![\pi]\!]t\}$
$[\![\varphi?]\!]$	$=$	$\xrightarrow{P?}$ where $P = \mathrm{cl}([\![\varphi]\!])$
$[\![\pi_1 \cup \pi_2]\!]$	$=$	$[\![\pi_1]\!] \cup [\![\pi_2]\!]$
$[\![\pi 1; \pi_2]\!]$	$=$	$\{(s, t) \mid \exists u, s[\![\pi_1]\!]u, u[\![\pi_2]\!]t\}$

NOTE: the negation of the language is classical (set-theoretic complement).

$\star\star$ *Probabilistic Logic for Quantum Programs*

Add to the language for quantum actions above formulas of the form $P^{\geq r}\varphi$. The lattices we have been working with abstracted away the numerical value of the inner product (which is where we get the probabilities from). Thus we define the semantics more directly on the Hilbert spaces. The focus on closed linear subspaces is most relevant to quantum tests. Such tests will make use of projections.

For a subset A of the Hilbert space $\mathcal{H}$, let $\mathrm{cl}(A)$ be the closed linear subspace generated by A. For a subset A and a vector v, let $\mathrm{Proj}_A\, v$ be the projection of v onto $\mathrm{cl}(A)$. Then if x is a one-dimensional subspace, $\mathrm{Proj}_A\, x = \{\mathrm{Proj}_A\, v \mid v \in x\}$ is a subspace.

Define a valuation V to map atomic propositions to closed linear subspaces of a Hilbert space. For a fixed Hilbert space and valuation, define $[\![\varphi]\!]$ as follows:

$[\![p]\!]$	$=$	$V(p)$
$[\![\neg\varphi]\!]$	$=$	$\{0\} \cup \mathcal{H} \setminus [\![\varphi]\!]$
$[\![\varphi_1 \wedge \varphi_2]\!]$	$=$	$[\![\varphi_1]\!] \cap [\![\varphi_2]\!]$
$[\![[\pi]\varphi]\!]$	$=$	$\{0\} \cup \{s \mid t \in [\![\varphi]\!]$ whenever $s[\![\pi]\!]t\}$
$[\![P^{\geq r}\varphi]\!]$	$=$	$\{0\} \cup \{s \neq 0 \mid \langle v, \mathrm{Proj}_{[\![\varphi]\!]}(v)\rangle \geq r$ where $v = s/\lVert s\rVert\}$
$[\![\varphi?]\!]$	$=$	$\{(s,t) \mid t = \mathrm{Proj}_{[\![\varphi]\!]}(s)\}$
$[\![\pi_1 \cup \pi_2]\!]$	$=$	$[\![\pi_1]\!] \cup [\![\pi_2]\!]$
$[\![\pi 1; \pi_2]\!]$	$=$	$\{(s,t) \mid \exists u, s[\![\pi_1]\!]u, u[\![\pi_2]\!]t\}$

A formula φ is satisfiable in a pair $(\mathcal{H}, V)$, where $\mathcal{H}$ is a Hilbert space and V is a valuation, if there exists a one dimensional subspace contained in $[\![\varphi]\!]$.

Problem 1: Completeness of Orthologic

Prove the completeness of orthologic with respect to the orthoframe realizations of orthologic, that is show that if $T \models \varphi$, then $T \vdash \varphi$. Here is a suggestion about how to proceed. Let $\mathbb{K} = (X, \not\perp, P, V)$ be an alleged canonical model, where

1. X is the set of all consistent deductively closed sets of formulas (X is consistent, and if $X \vdash \varphi$, then $\varphi \in X$). [Note that such sets are not necessarily maximal consistent sets (as used in the standard canonical model construction for the completeness of standard modal logic).]

2. For $T_1, T_2 \in X$, $T_1 \not\perp T_2$ iff for all φ, $T_1 \vdash \varphi$ implies $T_2 \nvdash \neg\varphi$.

3. P is the collection of sets $\mathcal{S} \subseteq X$, such that

$$T \in \mathcal{S} \Leftrightarrow [\forall U \in X((T \not\perp U) \Rightarrow \exists V(U \not\perp V \;\&\; V \in \mathcal{S}))]$$

   [Once $(X, \not\perp)$ is confirmed to be a non-orthogonality orthoframe, then P would be the set of *all* bi-orthogonally closed sets of $(X, \not\perp)$.]

4. $V(p) = \{T \in X \mid p \in T\}$

Show that

1. $\mathbb{K}$ is indeed an orthoframe realization of orthologic.

2. For all φ and for all $T \in X$, $T \models \varphi$ if and only if $\varphi \in T$

3. For each formula φ and for all sets of formulas T (not necessarily in X), if $T \models \varphi$, then $T \vdash \varphi$.

Problem 2: About properties of Piron lattices

Do one of the following:

1. Show that in an orthomodular lattice, the following are equivalent:
   (a) If a is an atom and $a \not\leq -p$ then $p \wedge (-p \vee a)$ is an atom
   (b) If a is an atom and if $a \wedge b = \mathbf{0}$, then $a \vee b$ covers b, that is if $b \leq p < a \vee b$ then $b = p$.

2. Show that in a propositional system $\mathbb{L}$, the following are equivalent:
   (a) If a, b are atoms, then there exists an atom c, such that $a \vee c = b \vee c = a \vee b$.
   (b) The lattice $\mathbb{L}$ is irreducible: $\mathbb{L}$ is not a direct sum of two sub lattices each containing at least two elements.

Problem 3: Logic for quantum actions

Given a quantum Kripke model $\mathfrak{M} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}}, V)$, let $\to$ be a subset of $\Sigma \times \Sigma$, such that $a \to b$ if and only if there is a $P \in \mathcal{L}$, such that $a \xrightarrow{P?} b$. Intuitively $\to$ is essentially the non-orthogonality relation $\not\perp$.

Let $\Box\varphi := [\neg\varphi?]\bot$ (where $\bot$ is a contradictory formula, such as $p \wedge \neg p$). Show that $s \in [\![\Box\varphi]\!]$ if and only if $t \in [\![\varphi]\!]$ for all t such that $s \to t$. (In other words, $\Box$ is the box modal operator for $\to$.)

Problem 4: Probabilistic logic for quantum actions

Do both of the following:

1. Given an example of a satisfiable formula of the form $P^{\geq r}\varphi \wedge P^{\geq s}\neg\varphi$, where $0 \leq r, s \leq 1$ and $r + s > 1$.
2. Show that for every Hilbert space and valuation $(\mathcal{H}, V)$, $[\![P^{\geq r}\varphi \to \neg Pr^{\geq(1-r+\epsilon)}[\varphi?]\bot]\!] = \mathcal{H}$ for every $\epsilon > 0$.

References

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