Quantum Logic: Homework 2

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Objective

The objective of this homework is to develop an understanding of the lattice properties of Hilbert spaces as well as logics for ortholattices and orthomodular lattices.

Preliminaries

$\star\star$ *Hilbert Quantum Logic*

The language of propositional Hilbert quantum logic is just the same as for classical propositional logic:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi$$

where $p \in \mathsf{AtProp}$ is a set of atomic proposition letters. We have the following abbreviations.

- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \to \psi := \neg\varphi \vee \psi$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$

Let V map each atomic in AtProp to a closed linear subspace of a fixed Hilbert space $\mathcal{H}$.[1] Formulas are interpreted via a function $[\![\cdot]\!]^V$ which maps each formula to a closed linear subspace of a fixed Hilbert space with the following constraints:

- $[\![p]\!]_V = V(p)$.
- $[\![\neg\varphi]\!]_V := [\![\varphi]\!]_V^\perp$
- $[\![\varphi \wedge \psi]\!]_V := [\![\varphi]\!]_V \cap [\![\psi]\!]_V$
- $[\![\varphi \vee \psi]\!]_V := ([\![\varphi]\!]_V^\perp \cap [\![\psi]\!]_V^\perp)^\perp$

An assignment V *weakly satisfies* a formula φ if $[\![\varphi]\!]_V \neq 0$, that is $[\![\varphi]\!]_V$ is not the 0-dimensional Hilbert space with just the element 0. As assignment V *strongly satisfies* a formula φ if $[\![\varphi]\!]_V = \mathcal{H}$. A formula φ is weakly (strongly) satisfiable if there exists an assignment V that weakly (strongly) satisfies φ. Note that if $\mathcal{H}$ is one-dimensional, then strong and weak satisfiability coincide, and we just say that φ is satisfiable.

[1] If $\mathcal{H}$ is finite dimensional, the algebraic structure that the atoms (and other formulas) are interpreted over may be described in terms of the *Grassmanian* (a topological structure consisting of the k-dimensional subspaces of a Hilbert space) of each dimension k or in terms of *projective geometries* (the "points" of the projective geometry are the "lines" or one-dimensional subspaces of a vector space).

$\star\star$ *Ortholattice, Orthoframe, Orthologic*

Ortholattice and orthologic

A partially ordered set $(A, \leq)$ is a *lattice* if for any two elements $a, b \in A$, there is a greatest lower bound $a \wedge b \in A$ and a least upper bound $a \vee b \in A$. A lattice is called *bounded* if there is a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$.

An *ortholattice* is a tuple $(A, \leq, -)$, such that $(A, \leq)$ is a (bounded) lattice with a maximum element $\mathbf{1}$ and a minimum element $\mathbf{0}$, and $-$ is a function from A to A that satisfies the following properties:

- $-a \vee a = \mathbf{1}$ and $-a \wedge a = \mathbf{0}$ for each $a \in A$
- $-(-a) = a$ for each $a \in A$
- $a \leq b$ implies $-b \leq -a$ for each $a, b \in A$

The language of orthologic is the same as for classical propositional logic.

An *ortholattice realization* of orthologic is a pair $\mathbb{L} = (L, V)$ where $L = (A, \leq, -)$ is an ortholattice and $V : \Phi \to L$ is valuation function mapping atomic proposition letters to elements of the lattice. The valuation function V extends to all formulas using the following conditions

1. $V(\neg\varphi) = -V(\varphi)$
2. $V(\varphi \wedge \psi) = V(\varphi) \wedge V(\psi)$.

We write $\mathbb{L} \models \varphi$ if $\mathbb{L} = (L, V)$ and $V(\varphi) = \mathbf{1}$.

Orthoframe and orthologic

An *orthogonality orthoframe* is a tuple (X, R), such that X is a set, and $R \subseteq X \times X$ is a relation satisfying

1. for no a does it hold that aRa (R is irreflexive)
2. if aRb then bRa (R is symmetric)

A *non-orthogonality orthoframe* is a tuple (X, R), such that X is a set, and $R \subseteq X \times X$ is a relation satisfying

1. aRa for each $a \in A$ (R is reflexive)
2. if aRb then bRa (R is symmetric)

Orthogonality orthoframes are called orthoframes in [5]. Non-orthogonality orthoframes are called orthoframes in [3, 4] and are called B-frames in [5] (B stands for Bouwerian). These two types of frames are closely related to each other in the following way. Given an orthogonality (non-orthogonality) orthoframe (X, R), the frame $(X, X \times X \setminus R)$ is a non-orthogonality (orthogonality) orthoframe (it is easy to check that this is true). We will focus on non-orthogonality orthoframes, and refer to non-orthogonality orthoframes simply as orthoframes.

Given an orthoframe (X, R) (with R reflexive), let $\perp = X \times X \setminus R$. We think of $\perp$ as an orthogonality relation (related any two states that are orthogonal), and for clarity we sometimes write $\not\perp$ for R, the non-orthogonality relation (accessibility via measurement). Furthermore, for any set $S \subseteq X$ and element $a \in X$, let $a \perp S$ iff $a \perp b$ for every $b \in S$, and let $S \perp a$ iff $S \perp a$. Finally, given sets $S, T \subseteq X$, we say $S \perp T$ iff $a \perp T$ for every $a \in S$.

A set $S \subseteq X$ is called a *proposition* of an orthoframe $F = (X, \not\perp)$ (also called *closed* in X) if for all $a \in X$,

$$a \in S \Leftrightarrow [\forall b(a \not\perp b \rightarrow \exists c(b \not\perp c \wedge c \in S))]$$

The left to right direction may seem a bit trivial (take $c = a$ and appeal to symmetry of $\not\perp$). But the right to left direction tells us that S is essentially closed under bi-orthogonality, that is $S = (S^{\perp})^{\perp}$.

An *orthoframe realization* of orthologic is a tuple $(X, \not\perp, P, V)$, where

1. $(X, \not\perp)$ is an orthoframe
2. P is a set of propositions that includes X, $\emptyset$, and is closed under orthocomplement $\perp$ and set theoretic intersection.
3. $V : \Phi \rightarrow P$ is a valuation function mapping each atomic proposition letter to a proposition in the othoframe.

The valuation function V can be extended to all formulas by:

1. $V(\neg\varphi) := V(\varphi)^{\perp} = \{a \in X \mid a \perp V(\varphi)\}$.
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$.

We write $\mathbb{F} \models \varphi$ if $\mathbb{F} = (X, \not\perp, P, V)$ and $V(\varphi) = X$.

$\star\star$ *Orthomodular quantum logic*

The language of orthomodular quantum logic is the same as for classical propositional logic, propositional Hilbert quantum logic, and orthologic.

An *algebraic realization* of orthomodular quantum logic is a pair $\mathbb{L} = (L, V)$ such that $\mathbb{L}$ is an ortholattice realization of orthologic and L is an orthomodular lattice.

A *Kripkean realization* of orthomodular quantum logic is a tuple $\mathbb{K} = (X, \not\perp, P, V)$, such that $\mathbb{K}$ is a orthoframe realization of orthologic and for every $a, b \in P$,

$$a \not\subseteq b \Rightarrow a \cap (a \cap b)^{\perp} \neq \emptyset.$$

$\star\star$ *Complexity theory*

There are many equivalent representations of a Turing machine. See, for example, [2] and [1]. But these all are equivalent in complexity. There are numerous complexity classes involving Turing machines (or non-deterministic Turing machines), such as P (polynomial time), NP (non-deterministic polynomial time), and PSPACE (polynomial space). The time and space complexity classes are comparable; in particular NP $\subseteq$ PSPACE, as the amount of space used (the number of tape entries used) cannot exceed the amount of time used. Given a complexity class $\mathcal{C}$, we say that a problem D is $\mathcal{C}$*-hard* if D is among the hardest problems of $\mathcal{C}$, that is there is a reduction of any problem in $\mathcal{C}$ to D. For P and NP problems, such reductions are taken to be many-one polynomial reductions, that is one problem is many-one reducible to another problem if there exists a function computable in polynomial time, such that converts the input of one problem to the input of the other problem in such a way that the outputs of the problem will be preserved. A problem D is $\mathcal{C}$*-complete* if D is in $\mathcal{C}$ and D is $\mathcal{C}$-hard. Here is an important result concerning the complexity of classical propositional logic.

Theorem 0.1 (Cook-Levin)**.** The problem of determining for any propositional formula φ whether it is satisfiable (classical propositional satisfiability) is NP-complete.

For comparison, satisfiability of basic modal logic is PSPACE-complete.

As many problems involve real and complex numbers, a model of computation that involves real or complex numbers is more appropriate than the Turing machine, which involves discrete entities. One such model of computation is the Blum-Shub-Smale machine, written BSS machine (see [8, Definition 2.1] for a definition followed by an explanation). One instruction used by a BSS machine is to assign a register a constant; BSS machines that do not involve constants are of special interest to us. The class of problems that can be solved by a BSS machine in polynomial time is written $\mathrm{P}_{\mathbb{R}}$, and there is a non-deterministic counterpart $\mathrm{NP}_{\mathbb{R}}$. The class of problems that can be solved by a non-deterministic polynomial time BSS machine that does not have constants and only allows for binary inputs is written $\mathrm{BP}(\mathrm{NP}_{\mathbb{R}})$.

Problem 1: Complexity of Hilbert Quantum Logic

Turn in one of the following:

1. Show that satisfiability of propositional Hilbert quantum logic in a one-dimensional Hilbert space is NP-complete. [You may use the Cook-Levin Theorem.]

2. Show that both weak and strong satisfiability of propositional Hilbert quantum logic in a two-dimensional Hilbert space over a field $\mathbb{F} \subseteq \mathbb{C}$ is NP-complete.

3. Show that both weak and strong satisfiability of propositional Hilbert quantum logic in a three-dimensional Hilbert space over $\mathbb{C}$ is in $\mathrm{BP}(\mathrm{NP}_{\mathbb{R}})$. [This problem is actually $\mathrm{BP}(\mathrm{NP}_{\mathbb{R}})$; but you are only asked show it is in this complexity class.]

Problem 2: Ortholattices, Modularity, and Orthomodularity

Turn in one of the following:

1. Prove that a modular ortholattice is an orthomodular lattice. Also, show that the lattice of closed linear subspaces of a Hilbert space is always orthomodular, but is modular if and only if it is finite dimensional. [Hint: You may use the fact that an ortholattice is modular if and only if it does not contain a "pentagon lattice" as a sublattice.]

2. Prove the equivalence of the following characterizations of the orthomodular law (you may assume the de Morgan laws hold):

   (a) $a \leq b$ implies $a \vee (-a \wedge b) = b$,

   (b) $a \leq b$ implies $b \wedge (-b \vee a) = a$,

   (c) $a \wedge (-a \vee (a \wedge b)) \leq b$.

   (d) $a \leq b$ if and only if $a \wedge -(a \wedge b) = \mathbf{0}$

   (e) ($a \leq b$ and $b \wedge -a = \mathbf{0}$) implies $a = b$

3. Prove the de Morgan laws hold for any ortholattice:

   (a) $-(a \wedge b) = -a \vee -b$

   (b) $-(a \vee b) = -a \wedge -b$.

Problem 3: Properties of Hilbert Lattices

Turn in one of the following:

1. Prove or disprove the validity of the following formulas (a formula φ is valid if $[\![\varphi]\!]^V = \mathcal{H}$ for every valuation V).
   (a) $p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$
   (b) $(p \wedge q) \vee (p \wedge r) \to p \wedge (q \vee r)$
   (c) $p \vee (q \wedge r) \to (p \vee q) \wedge (p \vee r)$
   (d) $(p \vee q) \wedge (p \vee r) \to p \vee (q \wedge r)$
2. Do both of the following:
   (a) Show that the lattice of closed linear subspaces of a Hilbert space over the complex numbers is complete (the lattice includes infinite meets and joins) and atomic.
   (b) Show that in any complete orthomodular lattice, the lattice is atomic if and only if it is atomistic.
3. Show that the lattice of closed linear subspaces of an infinite dimensional Hilbert space $\mathcal{H}$ over the complex numbers satisfies Mayet's condition: There is an ortholattice automorphism T (a bijection that preserves meets and orthocomplements) that maps closed linear subspaces onto closed linear subspaces, such that
   (a) There is a closed linear subspace Y, such that $T(Y) \subsetneq Y$.
   (b) There is a closed linear subspace Y, such that there are at least two distinct subspaces Z_1 and Z_2, such that $Z_1 \subsetneq Y$ and $Z_2 \subsetneq Y$ and for all closed subspaces $Z \subseteq Y$, $T(Z) = Z$.

Problem 4: Relating ortholattices to Kripke frames

Turn in both of the following:

1. Given an lattice realization of orthologic $\mathbb{L} = (A, \leq, -, V^L)$, let $K^{\mathbb{L}} = (X, \not\perp, P, V^K)$ be given by
   (a) $X = A \setminus \{\mathbf{0}\}$.
   (b) $a \not\perp b$ iff $a \not\leq -b$
   (c) $P = \{\{x \in X \mid x \leq a\} \mid a \in A\}$
   (d) $V^K(p) = \{b \in X \mid b \leq V^L(p)\}$

   Show the following:

   (a) $K^{\mathbb{L}}$ is an orthoframe realization of orthologic
   (b) for every φ, $\mathbb{L} \models \varphi$ if and only if $K^{\mathbb{L}} \models \varphi$
   (c) If $\mathbb{L}$ is an algebraic realization of orthomodular quantum logic, then $K^{\mathbb{L}}$ is a Kripkean realization of orthomodular quantum logic.
2. Given an orthoframe realization $\mathbb{F} = (X, \not\perp, P, V^K)$, let $L^{\mathbb{K}} = (A, \leq, -, V^L)$ be given by
   (a) $A := P$

(b) $a \leq b$ iff $a \subseteq b$ for each $a, b \in A$

(c) $-a := \{b \in X \mid a \perp b\}$

(d) $V^L(p) := V^K(p)$.

Show the following

(a) $L^{\mathbb{K}}$ is an ortholattice realization of orthologic

(b) for every φ, $\mathbb{K} \models \varphi$ if and only if $L^{\mathbb{K}} \models \varphi$.

(c) If $\mathbb{K}$ is a Kripkean realization of orthomodular quantum logic, then $L^{\mathbb{K}}$ is an algebraic realization of orthomodular quantum logic.

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