Quantum Logic: Homework 1

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Objective

The objective of this homework is obtain an appreciation for the mathematical and physical foundations of quantum logic, and gain a familiarity with the decidability problems concerning Hilbert quantum logic.

Preliminaries

$\star\star$ *Inner product space and Hilbert space*

A *vector space* over a field F is a tuple $V = (X, +, \cdot)$, such that

1. $(X, +)$ is an Abelian group, that is $+ : X \times X \to X$ is such that
   (a) $(x + y) + z = x + (y + z)$ (*associativity*)
   (b) $x + y = y + x$ (*commutativity*)
   (c) there exists 0, such that for all x, $x + 0 = 0 + x = x$ (*identity*)
   (d) for every x there exists a y, such that $x + y = y + x = 0$ (*inverses*)
2. $\cdot : F \times X \to X$ is given by (where $\cdot$ binds more tightly than $+$, and $\cdot$ is often omitted as is commonly done with multiplication)
   (a) $(a + b) \cdot x = a \cdot x + b \cdot x$
   (b) $a \cdot (x + y) = a \cdot x + a \cdot y$
   (c) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$
   (d) $1 \cdot x = x$ (where 1 is the multiplicative identity of F)

An *inner product space* is a tuple $(X, +, \cdot, \langle -, - \rangle)$, where $(X, +, \cdot)$ is a vector space, and $\langle -, - \rangle$ is a function from $X \times X \to F$ (for $F \in \{\mathbb{C}, \mathbb{R}\}$), called an *inner product*, having the following properties

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (*Conjugate symmetry*), where

   $\overline{z}$ is the complex conjugate of z, that is $\overline{a + bi} = a - bi$.
2. $\langle x, cy + z \rangle = c\langle x, y \rangle + \langle x, z \rangle$ (*Linearity in second*[1] *coordinate*)
3. $\langle x, x \rangle \geq 0$ for all x (*Non-negativity*)[2]
4. $\langle x, x \rangle = 0$ if and only if $x = 0$ (*Positive definiteness*)

[1]The literature is split between inner products defined with linearity in the first coordinate and linearity in the second coordinate. These are just local conventions, and otherwise the difference between these approaches is not important.

[2]Note that because of conjugate symmetry, $\langle x, x \rangle = \overline{\langle x, x \rangle}$, and hence $\langle x, x \rangle \in \mathbb{R}$.

Given an inner product $\langle -, - \rangle$, we define

- $\|x\| = \sqrt{\langle x, x \rangle}$
- $\mu(x, y) = \|x - y\|$

  (expanding this, we have $\mu(x, y) = \sqrt{\|x\|^2 + \|y\|^2 - 2\,\mathrm{re}\langle x, y \rangle}$, where $\mathrm{re}(a + bi) = a$ is the real part of $a + bi$).

One can check that $\|x\|$ is a *norm* and that μ is a *metric* (see [5, Chapter 1] for the definition of a norm and a metric). Any metric μ gives rise to a topology, whose open sets are generated by $\{y \mid \mu(x, y) < r\}$ for each x in the space and real number $r > 0$ (see [5, Chapter 1] for such a topology, called a *metric space*). A *Hilbert space* is an inner product space whose induced metric is *complete* (a metric is complete if every Cauchy sequence converges; see [5, Chapter 1] for Cauchy sequences and convergence in metric spaces).

Orthogonality, Orthonormal sets, orthonormal bases

For any subset W of an inner product space, The set *orthogonal* to W is $W^\perp = \{x \mid \langle x, y \rangle = 0, \text{ for all } y \in W\}$.

Given an inner product space $\mathcal{H}$, a set S of vectors is *orthonormal* if both of the following hold

1. Each vector $v \in S$ is a *unit vector* (that is $\|v\| = 1$).
2. Each pair of vectors $v, w \in S$ is *orthogonal* (that is $\langle v, w \rangle = 0$).

Given a set S, the (finite) span of S is the set

$$\mathrm{span}(S) = \{c_1 v_1 + \cdots + c_n v_n \mid n \in \mathbb{N}, c_1, \ldots, c_n \in F, v_1, \ldots, v_n \in S\}.$$

An *orthonormal basis* of an inner product space $\mathcal{H}$ is an orthonormal set S, such that $\mathcal{H} = \overline{\mathrm{span}(S)}$, where for any set T, $\overline{T}$ is the *closure* of T, that is, the smallest closed set containing T. Every basis has the same cardinality. The dimension of an inner product space is the cardinality of its basis. When the dimension of an inner product space is finite, then the span of the basis coincides with space itself.

An *isomorphism* between an inner product space $\mathcal{H}_1$ and $\mathcal{H}_2$ is a surjective linear function $f : \mathcal{H}_1 \to \mathcal{H}_2$, such that for every pair of vectors $x, y \in \mathcal{H}_1$, $\langle x, y \rangle = \langle f(x), f(y) \rangle$.

Theorem 0.1. Any two inner product spaces with the same dimension are isomorphic. Further more, any finite dimensional inner product space is a Hilbert space.

⋆⋆ *Projectors and Unitaries*

We assume throughout this section a finite dimensional Hilbert space. Quantum logic gates (basic processes of a quantum program) are generally composed of projectors and unitaries. For any linear map A, there exists a map $A^\dagger$, called its *adjoint*, with the property that for any vectors v and w, it holds that $\langle v, Aw \rangle = \langle A^\dagger v, w \rangle$. If A is represented using a complex valued matrix, then $A^\dagger$ is the complex transpose of A (the rows are switched with the columns, and then complex conjugate is taken of each of the entries in the matrix). Please see [6, Section 2.1.6] for more details.

A *Projector* is a linear map A with the following properties:

1. $A = A^\dagger$ ("self-adjoint" or "Hermitian")

2. $A \circ A = A$ (idempotent)

Projectors reflect the result of a successful quantum test of the subspace being projected onto. Whether such a test is successful is typically probabilistic, but assuming it is successful, the projector reflects the proper transformation of a quantum state.

A *Unitary operator* is a linear map for which $AA^\dagger = I$ (the adjoint is its inverse). While projectors are typically irreversible operations, unitary operations are reversible, and can be thought of as a change of basis.

$\star\star$ *Wave equation, Schrödinger equation, wave function*

A *wave equation* has the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where $u = u(x_1, \ldots, u_n, t)$ is a function of spatial parameters (position or momentum) x_i and time t, and u could represent the displacement, mass, velocity.

A certain type of wave equation is the free non-relativistic *Schrödinger equation*, given by:

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

where t is a time variable, x is a position variable, $\hbar = h/(2\pi)$ is the reduced Plank constant (with h being the Plank constant), and m is the mass. Any solution ψ to the Schrödinger equation is called a *wave function*. It has often taken to be the *state* of a quantum system (a complete description of the the quantum system). See [3, Chapter 4] for a discussion about wave functions and their relationship with Hilbert spaces.

$\star\star$ *L^2 space*

Let (X, μ) be a measure space, and F a field. The $L^2(X)$ space is a tuple $(Z, \langle \cdot, \cdot \rangle)$, where

- Z is the set of square integrable functions on X
- $\langle \cdot, \cdot \rangle : Z \times Z \to F$ is the L^2 inner product given by

  $$\langle f, g \rangle = \int_X \bar{f} g d\mu,$$

  where $\bar{g}$ is

  - the complex conjugate of g (mapping x to the complex conjugate $\overline{g(x)}$ of $g(x)$ for each x) if $F = \mathbb{C}$.
  - just g if $F = \mathbb{R}$.

Problem 1: Hilbert Spaces

Do three of the following parts:

1. (a) Explain why an L^2 space is a natural space for wave functions.
   (b) Show that the space $L^2(\mathbb{R})$ is a Hilbert space.
2. Show that for any finite-dimensional Hilbert space, each linear subspace is closed (in the topology induced by the metric induced by the inner product). (You may use Theorem 0.1)

3. Show, for any subset W of a Hilbert space (not necessarily a subspace), that $W^\perp$ is a closed linear subspace.

4. Show that for any closed linear subspace W of a Hilbert space, there exists a set Z, such that $W = Z^\perp$. (You may assume the Projection Theorem [5, Theorem 6.13]) (Hint: Consider $Z = W^\perp$)

5. Show that for any non-empty subset W of a Hilbert space (not necessarily a subspace), and any vector x in the Hilbert space, there exists $y \in W^\perp$ and $z \in (W^\perp)^\perp$, such that $x = y + z$. (In other words, $\mathcal{H} = W^\perp + (W^\perp)^\perp$). (You may assume the Projection Theorem [5, Theorem 6.13])

6. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ be a matrix representation for a linear map on an ordered basis (b_1, b_2, b_3). Let $B = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$ be a matrix representation for a linear map on an ordered basis (c_1, c_2). Give a matrix representation for tensor product $A \otimes B$, and describe what ordered basis is being used.

7. Show that the operators represented by the following matrices are unitary operators.

   - The *Hadamard transform*: $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
   - *Controlled not*: $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
   - *Pauly-Y transform*: $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Problem 2: Decidability and reduction

Using the translation described in [2], translate $\forall x \forall y (\neg(x \wedge y) = \neg x \vee \neg y)$ into an equivalent first-order formula over the complex numbers.

References

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