Reasoning with Probabilities
Mixed Strategies and Puzzles

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Mixed Strategies in Games

Game

Definition

A *game* is a tuple

$$\mathcal{G} = (I, \{\Pi_i\}_{i \in I}, \{u_i\}_{i \in I}),$$

$\mathcal{G}$ where

- I is a finite set of players,
- Π_i is a set of *pure strategies* for agent $i \in I$, and
- $u_i : \Pi \to \mathbb{R}$ is a utility function, assigning agent i's payoff to each pure strategy profile, (a pure strategy profile is a tuple $(\pi_1, \ldots, \pi_n)$, such that each $\pi_i \in \Pi_i$.)

Mixed strategies

Definition (Mixed strategy)

Given a set Π_i of pure strategies for a player i, a *mixed strategy* is a probability mass function

$$\sigma_i : \Pi_i \to [0,1],$$

that is, a function satisfying

$$\sum_{\pi_i \in \Pi_i} \sigma_i(\pi_i) = 1.$$

Let Σ_i be the set of all mixed strategies for player i.

Matching pennies example

- $I = \{a, b\}$
- $\Pi_i = \{H_i, T_i\}$ (heads and tails of player i's coin)
- $u_i : \Pi \to \{-1, 1\}$ is given by the following chart:

	H_b	T_b
H_a	$+1, -1$	$-1, +1$
T_a	$-1, +1$	$+1, -1$

Mixed strategies

Definition (Mixed profile)

A *mixed (strategy) profile tuple* is a tuple $(\sigma_1, \ldots, \sigma_n)$ for each player $i \in I$.

Definition (mixed strategy function)

A *mixed strategy function* is a function $\sigma : \Pi \to [0, 1]$, where Π is the set of pure strategy profiles.

Given a mixed profile $(\sigma_1, \ldots, \sigma_n)$, we can define a mixed strategy function $\sigma : \Pi \to [0, 1]$ by

$$\sigma(\pi) = \prod_{i=1}^{n} \sigma_i(\pi_i).$$

The original mixed profile can be recovered by

$$\sigma_i(\pi_i) = \sum_{\{\rho \in \Pi | \rho_i = \pi_i\}} \sigma(\rho).$$

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Correlated profiles

There exist mixed strategy functions that are now equivalent to mixed profiles: correlated strategies.

Example

σ	H_b	T_b
H_a	0.2	0.2
T_a	0.2	0.4

Here, whether a chooses H_a with probability $1/2$ or probability $1/3$ depends on b's strategy.

We will call a *mixed strategy function* a *mixed profile* only if the mixed strategy function is equivalent to a mixed profile (and are hence uncorrelated).

Expected utility

The utility function $u_i : \Pi \to \mathbb{R}$ can be extended from pure to mixed strategy profiles by

$$u_i(\sigma) = \sum_{\pi \in \Pi} \sigma(\pi) u_i(\pi).$$

Nash equilibrium

Given a mixed strategy profile σ and a mixed strategy ρ_i, denote by (ρ_i, σ_{-i}) the strategy profile

$$(\sigma_1, \ldots, \sigma_{i-1}, \rho_i, \sigma_{i+1}, \ldots, \rho_n).$$

Definition

A *Nash equilibrium* is a (mixed or pure) strategy profile σ such that for any agent i and any strategy ρ_i

$$u_i(\sigma) \geq u_i(\rho_i, \sigma_{-i}).$$

Nash equilibria in matching pennies

	H_b	T_b
H_a	$+1, -1$	$-1, +1$
T_a	$-1, +1$	$+1, -1$

- Pure strategy Nash equilibria: none
- Mixed strategy Nash equilibria: each player plays $1/2$ for both their strategies.
  [Corresponding collated profile: uniform probability function (each of the 4 pure profiles gets probability probability $\frac{1}{4}$)]

Background on some game logics

To give a sense of the diversity of game logics:

- Marc Pauly's dissertation, "Logic for social software", ILLC, University of Amsterdam, 2001.
  defines "Game logic" and "Coalition logic" with formulas describing what certain (groups of) agents can enforce.
- R. Alur, T. Henzinger, O. Kupferman. Alternating-Time temporal logic. Journal of the ACM. 2002.
  Describes powers of coalitions over time, using concurrent game models.
- J. Halpern: Substantive Rationality and Backward Induction. *Games and Economic Behavior*, 37:425-435, 2001.
  gives a logic for a fixed game that is defined on Kripke models whose states are labelled with (pure) strategy profiles.

Language

Given a game $\mathcal{G}$, let

$$t ::= aP(\pi) \mid t + t$$
$$\varphi ::= t \geq a \mid \neg\varphi \mid \varphi \wedge \varphi \mid [G]\varphi \mid [\sqsubset_i]\varphi \mid [\sqsupset_i]\varphi \mid [=_i]\varphi$$

where $a \in \mathbb{Q}$, $\pi \in \Pi$, $i \in I$, and $G \subseteq I$.

Formulas are evaluated on mixed strategy functions σ arising from mixed profiles

$\sigma \models \sum_{k=1}^{n} q_k P(\pi_k) \geq r$	iff	$\sum_{k=1}^{n} q_k \sigma(\pi_k) \geq r$
$\sigma \models [G]\varphi$	iff	$\tau \models \varphi$ whenever for each $i \in G$, $\sigma_i = \tau_i$
$\sigma \models [\sqsubset_i]\varphi$	iff	$\tau \models \varphi$ whenever $u_i(\sigma) < u_i(\tau)$
$\sigma \models [\sqsupset_i]\varphi$	iff	$\tau \models \varphi$ whenever $u_i(\sigma) > u_i(\tau)$
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Meaning of $[G]$

- $[G]\varphi$ means that φ is true for any strategy profile where those not in G potentially switch to different strategies.
- $[I \setminus \{i\}]\varphi$ means that φ is true whenever i potentially switches to a different strategy.
- $[\emptyset]\varphi$ means that φ is true is *all* profiles.

Abbreviations

Expressing that τ is the strategy profile:

$$\boldsymbol{\tau} \stackrel{\text{def}}{=} \bigwedge_{\pi \in \Pi} P(\boldsymbol{\pi}) = \tau(\pi)$$

The utility for i is term

$$\mathbf{u}_i \stackrel{\text{def}}{=} \sum_{\pi \in \Pi} u_i(\pi) P(\boldsymbol{\pi})$$

The probability i has for playing pure strategy π_i is a term:

$$P_i(\boldsymbol{\pi}_i) \stackrel{\text{def}}{=} \sum \{P(\boldsymbol{\rho}) \mid \rho \in \Pi, \rho_i = \pi_i\}$$

Expressing that τ_i is i's (mixed) strategy

$$\boldsymbol{\tau}_i \stackrel{\text{def}}{=} \bigwedge_{\pi_i \in \Pi_i} (P_i(\boldsymbol{\pi}_i) = \tau_i(\pi_i))$$

Expressing Nash Equilibrium

Definition (Best response)

Given a mixed strategy profile σ, i's strategy is a best response if for every formula φ, we have

$$\sigma \models \varphi \rightarrow [(I \setminus \{i\})]\langle \sqsubseteq_i \rangle \varphi.$$

Given a specific σ we define

$$\mathrm{br}_i(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma} \rightarrow [(I \setminus \{i\})]\langle \sqsubseteq_i \rangle \boldsymbol{\sigma}.$$

A Nash equilibrium is a mixed strategy profile, such that everyone's strategy is a best response. For each σ, define

$$\mathrm{Nash}(\boldsymbol{\sigma}) \equiv \bigwedge_{i \in I} \mathrm{br}_i(\boldsymbol{\sigma}).$$

So σ is a Nash equilibrium in $\mathcal{G}$ if and only if $\models \mathrm{Nash}(\boldsymbol{\sigma})$

Axioms

- Classical Logic Tautologies
- $[\star](\varphi \to \psi) \to ([\star]\varphi \to [\star]\psi)$, with $\star \in \{G, \sqsubset_i, \sqsupset_i, =_i\}$.
- $[*]\varphi \to \varphi$, with $* \in \{G, =_i\}$ (reflexivity)
- $[i]\varphi \to [G]\varphi$ $(i \in G)$
- $[\sqsubset_i]\varphi \to [=_i][\sqsubset_i]\varphi$
- $[\sqsupset_i]\varphi \to [=_i][\sqsupset_i]\varphi$
- $\pm P_i(\boldsymbol{\pi}_i) \geq q \to [i] \pm P_i(\boldsymbol{\pi}_i) \geq q$ ($[i]$ fixes i's strategy)
- $\pm \mathbf{u}_i \geq q \to [=_i] \pm \mathbf{u}_i \geq q$ ($[=_i]$ fixes i's utility)
- $\mathbf{u}_i \geq q \to [\sqsubset_i]\mathbf{u}_i > q$
- $\mathbf{u}_i \leq q \to [\sqsupset_i]\mathbf{u}_i < q$
- Global modality axioms (next slide)
- Probability Bounds and Restrictions (next slide)
- Inequality axioms (a later slide)
- Rules (a later slide)

Global modality axioms

Note that $[\emptyset]$ ("in all mixed profiles") serves as a global modality.

- $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\bigwedge_{i \in G} \boldsymbol{\sigma}_i \to \langle G \rangle \varphi)$
  (If φ is true at σ, then $\langle G \rangle \varphi$ is true in any τ that agrees with σ on the strategies of those in G.)
- $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\mathbf{u}_i = u_i(\sigma) \to \langle =_i \rangle \varphi)$
- $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\mathbf{u}_i < u_i(\sigma) \to \langle \sqsubset_i \rangle \varphi)$
- $[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\mathbf{u}_i > u_i(\sigma) \to \langle \sqsupset_i \rangle \varphi)$

Bound axioms

- $P(\boldsymbol{\pi}) \geq 0$
- $\sum_{\pi \in \Pi} P(\boldsymbol{\pi}) = 1$

We can restrict the domain of the model being characterized to $\widetilde{\Sigma} \subseteq \Sigma$

- $\neg \boldsymbol{\sigma}$ for each $\sigma \notin \widetilde{\Sigma}$

Inequality axioms

- (Permutation)
  $\sum_{k=1}^{n} q_k P(\pi_k) \geq q \rightarrow \sum_{k=1}^{n} q_{j_k} P(\pi_{j_k}) \geq q$
- (Adding and deleting zero terms)
  $t \geq q \leftrightarrow t + 0P(\pi_{k+1}) \geq q$
- (Adding coefficients)
  $\sum_{k=1}^{n} q_k P(\pi_k) \geq q \wedge \sum_{k=1}^{n} q'_k P(\pi_k) \geq q' \rightarrow \sum_{k=1}^{n} (q_k + q'_k) P(\pi_k) \geq (q + q')$
- (Multiplying my a non-negative constant)
  $t \geq q \leftrightarrow dt \geq dq$ where $d > 0$
- (Monotonicity)
  $(t \geq q) \rightarrow (t > q')$ where $q > q'$

Rules

Modus Ponens

$$\frac{A \vdash \varphi \to \psi \quad A \vdash \varphi}{A \vdash \psi}$$

Necessitation $\star \in \{G, =_i, \sqsubset_i, \sqsupset_i\}$

$$\frac{\vdash \varphi}{\vdash [\star]\varphi}$$

Monotonicity

$$\frac{\vdash \varphi}{A \vdash \varphi}$$

Pseudo modalities and another rule

Definition (Pseudo modalities)

Let $s_i \in \{G, \sqsubset_i, \sqsupset_i, =_i\} \cup \mathcal{L}$. Define $[(s_1, \ldots, s_n)]\varphi$ as follows

- $[()]\varphi \stackrel{\text{def}}{=} \varphi$
- $[(\psi, s_2, \ldots, s_n)]\varphi \stackrel{\text{def}}{=} \psi \to [(s_2, \ldots, s_n)]\varphi$
- $[(a, s_2, \ldots, s_n)]\varphi \stackrel{\text{def}}{=} [a][(s_2, \ldots, s_n)]\varphi$

For each $s = (s_1, \ldots, s_n)$

$$\frac{A \vdash [s](P(\pi) \neq q) \text{ for all } q \in \mathbb{Q}, q \neq p}{A \vdash [s](P(\pi) = p)}$$

Each pure profile must be assigned a probability by each agent.

$$\frac{\vdash \boldsymbol{\tau} \to \varphi \text{ for all } \tau \in \widetilde{\Sigma} \text{ such that } \tau_i = \sigma_i \text{ for all } i \in G}{\vdash \boldsymbol{\sigma} \to [G]\varphi}$$

(Compare with the axiom
$[\emptyset](\boldsymbol{\sigma} \to \varphi) \to [\emptyset](\bigwedge_{i \in G} \boldsymbol{\sigma}_i \to \langle G \rangle \varphi)$)

$$\frac{\vdash \boldsymbol{\tau} \to \varphi \text{ for all } \tau \in \widetilde{\Sigma} \text{ such that } u_i(\tau) = u_i(\sigma)}{\vdash \boldsymbol{\sigma} \to [=_i]\varphi}$$

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Examples and Puzzles

Lottery paradox

Suppose we want to define belief as having probability at least r for some $r < 1$, that is we fix a r and set

$$B(\varphi) \equiv P(\varphi) \geq p.$$

- Suppose the chance a lottery ticket does not win is r (say $99,999/100,000$).
- If p_i is the proposition that lottery ticket i looses the lottery, then $B(p)$ is true.
- Let n be the number of lottery tickets
- Then $\bigwedge_{i=1}^{n} B(p_i)$ is true.
- Normal modal operators distribute over conjunction: $B(\varphi \wedge \psi) \leftrightarrow B(\varphi) \wedge B(\psi)$
- Then as $B(\bigwedge_{i=1}^{n} p_i) \leftrightarrow \bigwedge_{i=1}^{n} B(p_i)$, you believe that all the tickets will loose.

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Cable Guy Paradox

A cable guy is coming to your home between 8 a.m. and 4 p.m., and you must be at home when he arrives. Unfortunately, you do not know when exactly he will come. Now, you place a bet with someone as to whether the cable guy will come during the time interval $(8, 12]$ or the time interval $(12, 16)$. Until 8 a.m., you consider both intervals equally appealing. But regardless of when the cable guy actually comes, there will some period of time after 8 a.m. and before his arrival, and during this period, the probability of his arriving in the morning is less than for his arriving in the afternoon.

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Monty Hall Puzzle

Suppose you are on a game show, and you are given the choice of three doors. Behind one door is a car, and behind the others are goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which is a goat. He says "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

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Two-envelop problem

Suppose you are presented with two envelopes, and are told that each contains money, and that one of them contains twice as much money as the other. You are asked to choose an envelop in which you can keep. You pick one of them, but before opening it, you are asked if you would like to switch. Is it to your advantage to switch?

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Analyzing the two-envelop problem

Let the amount of money in the envelop you selected be n. You might guess that there is a 50% chance that the other envelop has more money. Then the expected value of switching would be:

$$\frac{1}{2} \cdot 2n + \frac{1}{2} \cdot n/2 = \frac{5n}{4} > n.$$

Analysis of this problem suggests the importance of a prior probability on selecting n and the probability the one you chose having more being conditional on n.

See also,

- D. Samet, I. Samet and D. Shmeidler. One Observation behind Two- Envelope Puzzles.

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See also,

- D. Samet, I. Samet and D. Shmeidler. One Observation behind Two- Envelope Puzzles.

Analyzing the two-envelop problem

Let the amount of money in the envelop you selected be n. You might guess that there is a 50% chance that the other envelop has more money. Then the expected value of switching would be:

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General Remarks

- The involvement of σ-algebras:
  - non-discrete (Vitali set issues)
  - discrete, but allows us to represent greater uncertainty (qualitative uncertainty of non-measurable sets)
- Quantitative uncertainty vs qualitative uncertainty:
  - quantitative uncertainty is expressed using probability formulas $P_i(\varphi) \geq r$ and
  - qualitative uncertainty is expressed using modal operators $[i]\varphi$

  Mixing these allows us to represent qualitative uncertainty over probabilities.
- Conditioning vs updating:
  - Condition probability $P(\varphi \mid \psi) \geq r$: After learning ψ, then $P(\varphi \mid \psi)$ is the probability i gives to the truth of φ before learning ψ.
  - Updated probability $[\psi]P_i(\varphi) \geq r$ (where ψ is an action (A, e) of everyone learning ψ): After learning ψ, then $[\psi]P_i(\varphi)$ is what i gives to the truth of φ after learning ψ.