Reasoning with Probabilities
Mixing Qualitative and Quantitative

Joshua Sack

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Probabilistic Epistemic Logic

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Let AP be a set of proposition letters and I a set of agents.
Formulas:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [i]\varphi \mid t_i \geq r$$

where $p \in AP$, $r \in \mathbb{Q}$, and t_i is a term for agent i
Terms for $i \in I$:

$$t_i ::= aP_i(\varphi) \mid t_i + t_i$$

where $a \in \mathbb{Q}$.

- This language is from:
  R. Fagin & J. Halpern (1994) Reasoning about Knowledge and Probability. *Journal of the ACM* 41:2, pp. 340–367.

Qualitative and Quantatative

- $[i]\varphi$: Qualitative uncertainty by agent i regarding φ
- $P_i(\varphi) \geq r$: Quantitative uncertainty by agent i regarding φ

Probabilistic epistemic models and semantics

Let AP be a set of proposition letters and I a set of agents.
$M = (X, R, \|\cdot\|, \mathbb{P})$, where

- $(X, R, \|\cdot\|)$ is an epistemic model
- $\mathbb{P}$ is a function fram I to functions $\mathbb{P}_i$ mapping each state x to probability space $(S_{i,x}, \mathcal{A}_{i,x}, \mu_{i,x})$, such that $S_{i,x} \subseteq X$.

The semantics of formulas is defined by a function $[\![\cdot]\!]$ from formulas to subsets of X.

$$
\begin{array}{lcl}
[\![\top]\!] & = & X \\
[\![p]\!] & = & \|p\| \\
[\![\neg\varphi]\!] & = & X - [\![\varphi]\!] \\
[\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![[i]\varphi]\!] & = & l_i([\![\varphi]\!]) \\
[\![\sum_{k=1}^{n} a_k P_i(\varphi_k) \geq r]\!] & = & \{x \mid \sum_{k=1}^{n} a_k \mu_{i,x}([\![\varphi_k]\!] \cap S_{i,x}) \geq r\}
\end{array}
$$

Relating Belief and Probability

We ofter define belief in terms of probability:

$$[i]\varphi \equiv P_i(\varphi) = 1$$

Given a discrete probabilistic modal model $(X, \|\cdot\|, \{\mathbb{P}_i\}_{i\in I})$, we can define an epistemic relation R_i such that

$$xR_iy \text{ if and only if } \mathbb{P}_{i,x}(y) > 0$$

But if the probabilistic modal model is not discrete, we cannot necessarily define such a relation. We simply define $[i]$ directly in terms of probability.

If we define belief as such, there is no need for probabilistic epistemic models.

Motivation for separating qualitative and quantitative

What if we want to have qualitative uncertainty over what the probability distribution is?

An example illustrating such a situation is given in the next slide.

Fagin, Halpern, and Tuttle example

Suppose there are two agents i and k.

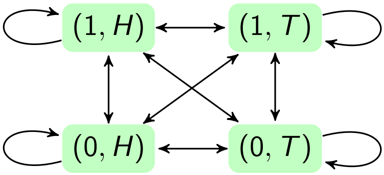
1. k is first given a bit 0 or 1. k learns he has this bit, i is aware that k received a bit, but i does not know what bit k received.
2. k flips a fair coin and looks at the result. i sees k look at the result, but does not what the result is.
3. k performs action s if the coin agrees with the bit (given that heads agrees with 1 and tails agrees with 0), and performs action d otherwise.

This example is from

- R. Fagin & J. Halpern (1994) Reasoning about Knowledge and Probability. *Journal of the ACM* 41:2, pp. 340–367.

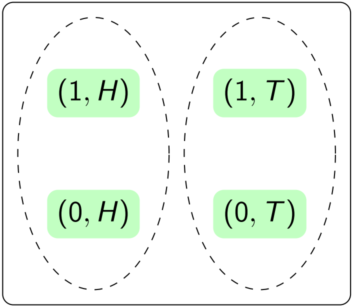
Discussion

There are four possible sequences of events: $(1, H), (1, T), (0, H), (0, T)$ (note that the action s or d is determined from the first two steps). Until k performs the action s or d, agent i considers any of these four states possible.



We indicate i's uncertainty between two states using a bidirectional arrow between the two states. In particular, an arrow from state x to state y indicates that i considers y possible if x is the actual state. (Before the bit is given, k's epistemic relation will be the same).

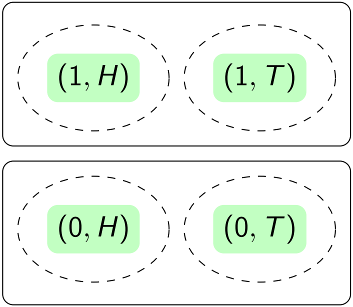
Here is a possibility for i's probability spaces. The sample space enclosed in a box, and the σ-algebra equivalence classes are enclosed in the dotted ovals.



$\mathbf{M}_1$

The sample space is the same as the set of states i considers possible. Individual states cannot be measurable (otherwise 0 or 1 must be assigned a probability).

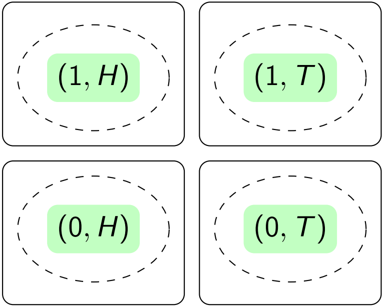
Another possibility has a sample space containing only the states with the correct bit (but recall that i considers all states possible and both sample spaces possible).



$\mathbf{M}_2$

Without assigning probability to the bit, i can now assign a probability to the actions s and d.

Here i is uncertain among 4 probability spaces.



$\mathbf{M}_3$

Mixing qualitative and quantitative

When mixing probability and epistemics, each represents beliefs about different aspects of a situation. In the previous example, there may be

1. quantitative (probability) beliefs about the coin toss
2. qualitative beliefs about the bit or about the probabilities themselves

Representing uncertainty about probabilities

- unmeasurable sets:
  - advantage of allowing us to clearly represent an agent's complete uncertainty about the probability of an situation.
  - disadvantage of excluding potentially reasonable sets from having a probability (such as the probability of $\{(H,1),(T,0)\}$, that is agent k doing action s).
- uncertainty about probabilities
  - advantage of allowing us to divide an unmeasurable set into subsets each in different probability spaces.
  - advantage of allowing us to reflect uncertainty between/among specific probability spaces.
  - disadvantage of requiring all probability measures considered possible be explicit; complete uncertainty requires all infinitely many possible probability measures.

Proof System for PEL

- All propositional tautologies
- $[i](\varphi \to \psi) \to ([i]\varphi \to [i]\psi)$
- $[i]\varphi \to \varphi$
- $[i]\varphi \to [i][i]\varphi$
- $\neg[i]\varphi \to [i]\neg[i]\varphi$
- $P_i(\varphi) \geq 0$
- $P_i(\top) = 1$
- $P_i(\varphi \wedge \psi) + P_i(\varphi \wedge \neg\psi) = P_i(\varphi)$
- Inequality axioms (Next slide)
- If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.
- If $\vdash \varphi$, then $\vdash [i]\varphi$.
- If $\vdash \varphi \leftrightarrow \psi$, then $\vdash P_i(\varphi) = P_i(\psi)$.

Inequality axioms

- (permutation)
  $a_1 P_i(\varphi_1) + \cdots + a_n P_i(\varphi_n) \geq r \to$
  $a_{j_1} P_i(\varphi_{j_1}) + \cdots + a_{j_n} P_i(\varphi_{j_n}) \geq r$
- (adding coefficients)
  $(\sum_{k=1}^{n} a_k P_i(\varphi_k) \geq r) \wedge (\sum_{k=1}^{n} b_k P_i(\varphi_k) \geq s) \to$
  $(\sum_{k=1}^{n} (a_k + b_k) P_i(\varphi_k) \geq (r + s))$
- (adding and deleting 0 terms)
  $(t \geq r) \leftrightarrow (t + 0P_i(\varphi) \geq r)$
- (multiplying by non-zero coefficient)
  $t \geq r \leftrightarrow at \geq ar$ whenever $a > 0$.
- (dichotomy)
  $t \geq r \vee t \leq r$
- (monotonicity)
  $t \geq r \to t > s$, whenever $r > s$.

Completeness

- Fix a consistent formula θ
- Let Δ be the set of subformulas and negations of subformulas of θ. (Δ is finite.)

$\mathcal{M} = (X, R, \|\cdot\|, \mathbb{P})$, where

- X is the set of maximally consistent subsets of Δ
- xR_iy iff for all $[i]\varphi \in \Delta$, $[i]\varphi \in x$ iff $[i]\varphi \in y$.
- $\|p\| = \{x \in X \mid p \in x\}$
- $\mathbb{P} = \{(S_{i,x}, \mathcal{A}_{i,x}, \mu_{i,x})\}$
    - $S_{i,x} = X$
    - $\mathcal{A}_{i,x} = \mathcal{P}(X)$
    - $\mu_{i,x}$ is any function satisfying conditions of next slides.

Lemma for Completeness

- $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be the set of subsets of θ,
- $At(\Sigma) = \{\bigwedge_{i=1}^{n} \delta_i \mid \delta_i \in \{\sigma_i, \neg\sigma_i\}\}$

Lemma

Let $t \geq r$ be a probability formula. Let $At(\Sigma) = \{\alpha_1, \ldots, \alpha_{2^n}\}$. Then there are rationals $a_1, \ldots, a_{2^n}$ such that $t \geq r$ is equivalent to $a_1 P_i(\alpha_1) + \cdots + a_{2^n} P_i(\alpha_{2^n}) \geq r$.

Let $At(\Sigma, \varphi) = \{\alpha \in At(\Sigma) \mid \vdash \alpha \to \varphi\}$. Then

$$P(\varphi) \equiv \sum_{\alpha \in At(\Sigma,\varphi)} P(\varphi \wedge \alpha) \equiv \sum_{\alpha \in At(\Sigma,\varphi)} P(\alpha).$$

The first equivalence comes from multiple applications of additivity for each subformula σ_i.

For each $x \in X$, let $\widehat{x} = \bigwedge_{\{\delta \in x\}} \delta$.
Note: $\{\widehat{x} \mid x \in X\} \subseteq At(\Sigma)$, and

$$\{\widehat{x} \mid \psi \in x\} = At(\Sigma, \psi) := \{\alpha \in At(\Sigma) \mid \vdash \alpha \to \varphi\}.$$

- Fix i and x.
- Let $\{t_1 \geq r_1, \ldots, t_k \geq r_k\}$ be the i inequality formulas in x.
- Let $\{t_{k+1} \geq r_{k+1}, \ldots, t_m \geq r_m\}$ be the i inequality formulas in $\Delta - x$.
- Each formula $t_j \geq r_j$ is equivalent to $a_{j,1} P_i(\alpha_1) + \cdots + a_{j,2^n} P_i(\alpha_{2^n}) \geq r_j$
- Each formula $t_j \geq r_j$ is equivalent to $\sum_{y \in X} a_{j,x} P_i(\widehat{y}) \geq r_j$

System of inequalities

Let $X = \{y_1, \ldots, y_\ell\}$. Let $\mu_{i,x}$ be defined on X as a solution to:

$$
\begin{array}{|rcl|}
\hline
\sum_{y\in X} a_{1,y}\mu_{i,x}(y) & \geq & r_1 \\
 & \vdots & \\
\sum_{y\in X} a_{k,y}\mu_{i,x}(y) & \geq & r_k \\
\hline
\sum_{y\in X} a_{k+1,y}\mu_{i,x}(y) & < & r_{k+1} \\
 & \vdots & \\
\sum_{y\in X} a_{m,y}\mu_{i,x}(y) & < & r_m \\
\hline
\sum_{y\in X} \mu_{i,x}(y) & \geq & 1 \\
-\sum_{y\in X} \mu_{i,x}(y) & \geq & -1 \\
\hline
\mu_{i,x}(y_1) & \geq & 0 \\
 & \vdots & \\
\mu_{i,x}(y_\ell) & \geq & 0 \\
\hline
\end{array}
$$

Completeness follows from a truth lemma:

Lemma

For every formula $\varphi \in \Delta$ *and state* $x \in X$,

$$\varphi \in x \textit{ iff } (M, x) \in [\![\varphi]\!]$$

- This is proved by induction on the structure of the formula, and is similar to the proof of the truth lemma for basic epistemic logic.
- Note that the case for probability formulas $t \geq r$ does not make use of the induction hypothesis, but follows directly from the choice of the probability measure.

Complexity lower bound

Satisfiability of epistemic logic is known to be PSPACE complete. As epistemic logic is a fragment of probabilistic epistemic logic (a very simple reduction to probabilistic epistemic logic), then probabilistic epistemic logic is PSPACE-hard

Complexity upper bound

The upper bound is also PSPACE.

Proposition

PSPACE = NPSPACE

One can non-deterministically construct a tableaux for an input φ with polynomial branching to test whether a formula is accepted by a tableaux. (Acceptance by a tableaux implies that φ is satisfiable.) Tableaux acceptance can be checked in PSPACE.

Probabilistic Automata

Probabilistic automaton

Let $Dist(S)$ be the set of all discrete probability distributions (or mass functions) on a set S.

Definition

A *probabilistic automaton* (augmented with a valuation) is a tuple $(S, I, \{\xrightarrow{i}\}_{i \in I}), \|\cdot\|)$, such that

- S is a set of states
- $\xrightarrow{i} \subseteq S \times Dist(S)$
- $\|\cdot\| : AP \to \mathcal{P}(S)$.

Two-sorted probability language

State formulas (with $i \in I$)

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\xrightarrow{i}]\psi$$

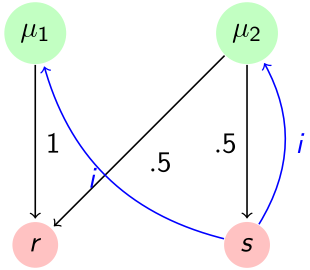
Distribution formulas (with $r \in \mathbb{Q}$)

$$\psi ::= \top \mid \neg\psi \mid \psi \wedge \psi \mid L_p(\varphi)$$

Semantics

$s \models p$	iff $s \in \|p\|$
$s \models T$	iff always
$\mu \models T$	iff always
$s \models \neg\varphi$	iff $s \not\models \varphi$
$\mu \models \neg\psi$	iff $\mu \not\models \psi$
$s \models \varphi_1 \wedge \varphi_2$	iff $s \models \varphi_1$ and $s \models \varphi_2$
$\mu \models \psi_1 \wedge \psi_2$	iff $\mu \models \psi_1$ and $\mu \models \psi_2$
$s \models [\xrightarrow{i}]\psi$	iff $\mu \models \psi$ for all μ such that $s \xrightarrow{i} \mu$
$\mu \models L_r\varphi$	iff $\mu(\{s \mid s \models \varphi\}) \geq r$

Example



$r \models [i]\neg\top$
$s \models [i]L_{.5}[i]\neg\top$
$s \models \langle i\rangle\neg L_1[i]\neg\top$
$s \models \langle i\rangle L_1[i]\neg\top$

Syntactic limitation of two-sorted language

The two-sorted language forbids certain higher-order constructs, such as

- $[a][b]p$
- $L_{.3}L_{.6}p$
- $L_{.3}(p \wedge L_{.6}q)$

Lifting and flattening

Definition (lifting)

Given a relation $R \subseteq X \times Y$, define a *lifting* $\ell(R) \subseteq Dist(X) \times Dist(Y)$ of R by

$$\mu \ell(R) \nu \Leftrightarrow (\forall A \subseteq X)(\mu(A) \leq \nu(R(A))).$$

Definition (flattening)

Given a $\mu \in Dist(Dist(S))$, define the *flattening* of μ by the function $\mathsf{fl} : Dist(Dist(S)) \to Dist(S)$ by

$$\mathsf{fl}(\mu)(s) = \sum_{\nu' \in \mathsf{supp}(\mu)} \mu(\nu')\nu'(s).$$

where $\mathsf{supp}(\mu)$ is the support of μ ($\{x \mid \mu(x) > 0\}$).

Defining over distributions

Definition ($\overset{i}{\rightarrowtail}$)

Given a transition $\overset{i}{\to} \subseteq S \times Dist(S)$, I let $\overset{i}{\rightarrowtail} \subseteq Dist(S) \times Dist(S)$ be defined by $\mu \overset{i}{\rightarrowtail} \nu$ if and only if there exists ν', such that $\mu \ell(\overset{i}{\to})\nu'$ and $\nu = \text{fl}(\nu')$.

Definition (Lifting of a measure)

Also given $\mu \in Dist(S)$ let $\breve{\mu}$, where

$$\breve{\mu}(\nu) = \begin{cases} \mu(s) & \nu = \delta_s \\ 0 & \text{otherwise} \end{cases}.$$

Lifting of probabilistic automata

Given a probabilistic automaton

$$\mathbb{A} = (S, I, \{\xrightarrow{i}\}_{i \in I}, \| \cdot \|),$$

define its lifting

$$Lift(\mathbb{A}) = (Dist(S), \{\overset{i}{\rightarrowtail}\}, V),$$

where $V : AP \to \mathcal{P}(Dist(S))$, such that $\mu \in V(p)$ if and only if $\text{supp}(\mu) \cap \|p\| \neq \emptyset$.

The exact definition of V here is somewhat arbitrary, but it relates to predicate liftings in coalgebraic modal logic.

One-sorted language and semantics

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\overset{i}{\rightarrowtail}]\varphi \mid L_r(\varphi)$$

Given a probabilistic automaton $\mathbb{A} = (S, I, \{\overset{i}{\to}\}_{i \in I}, \| \cdot \|)$ (with lifting $(Dist(S), \{\overset{i}{\rightarrowtail}\}, V)$)

$\mu \models p$	iff $\mu \in V(p)$
$\mu \models \neg\psi$	iff $\mu \not\models \psi$
$\mu \models \psi_1 \wedge \psi_2$	iff $\mu \models \psi_1$ and $\mu \models \psi_2$
$\mu \models [\overset{i}{\rightarrowtail}]\varphi$	iff $\nu \models \psi$ whenever $\mu \overset{i}{\rightarrowtail} \nu$
$\mu \models L_r\varphi$	iff $\breve{\mu}(\{\nu \mid \nu \models \varphi\}) \geq r$